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PERTURBING EVOLUTIONARY SYSTEMS BY STEP RESPONSES AND CUMULATIVE OUTPUTS

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Abstract. We develop a perturbation theory for strongly continuous backward evolutionary systems and their adjoint systems. The theory is not based on generating families but on certain operator families called step responses and cumulative outputs. The perturbation problem is reduced to solving an abstract Stieltjes integral equation of nonconvolution type. The theory is well suited for treating structured population models.

1. Introduction. Many mathematical models which are traditionally written as linear partial differential equations or functional differential equations, can alternatively and with advantage be formulated as abstract Cauchy problems

$$\begin{aligned} \frac{d}{dt}u(t) &= A(t)u(t), \quad t > s, \\ u(s) &= x, \end{aligned} \tag{1.1}$$

where the initial state x belongs to the Banach space X and $\{A(t)\}$ is a family of linear operators in X. In the autonomous case, $A = A(t)$ for all t and proving well-posedness of the problem (1.1) amounts to showing that the operator A generates a semigroup. This can be done for instance by employing the (generalized) Hille-Yosida theorem, which gives necessary and sufficient conditions for A to be the infinitesimal generator of a strongly continuous semigroup on X.

The time-dependent case is much more difficult than the autonomous one. There are known sufficient conditions guaranteeing that the family $\{A(t)\}$ generates (in a certain precise sense) an evolutionary system $\{U(t,s)\}$ on X. These conditions include the complicated Kato stability conditions and the requirements that for fixed t the operator $A(t)$ generates a strongly continuous semigroup and that the intersection of all domains $\mathcal{D}(A(t))$ is dense in X (see Kato (1953), Pazy (1983)). Another notion of generating family, together with interesting generation results, is given by Dorroh and Graff (1979). On the other hand, given an evolutionary system, there seems to be no way of defining a generating family that

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perturbations mapping into a super space and applied their theory to retarded functional-differential equations and age-dependent population dynamics. Their approach is, however, less well-suited for structured population problems with higher dimensional individual state space — the difficulty being the identification of the so-called sun-space $X^{\odot}$ and the domains of the infinitesimal generators.

One of the main messages of this paper is that the difficulties described above are sometimes due to our insistence on using differential equations as the language for formulating mathematical models. The derivation of a differential equation involves taking limits, which in the infinite dimensional case amounts to calculating domains of unbounded operators. This can sometimes be a formidable mathematical task that does not necessarily give any insight into the underlying physical problem. We shall therefore present a perturbation theory based on *integral* equations that all together avoids unbounded operators. This theory corresponds to a very natural approach to modeling structured populations, see Diekmann, Gyllenberg, Metz, and Thieme (1994).

To explain our ideas and to motivate our approach we start by doing some formal calculations. Assume that there is an evolutionary system U_0 somehow associated with the family $\{A_0(t)\}$ and that equation (1.5) has a unique solution given via a perturbed evolutionary system U, that is, $u(t) = U(t,s)x$. Then U should satisfy the *variation-of-constants formula*

$$U(\tau,s) = U_0(\tau,s) + \int_s^\tau U_0(\tau,\sigma)B(\sigma)U(\sigma,s)\,d\sigma. \tag{1.6}$$

If one multiplies formula (1.6) from the left by $B(\tau)$ and integrates from s to t with respect to τ one obtains

$$V(t,s) = V_0(t,s) + \int_s^t V_0(t,\sigma)V(d\sigma,s), \tag{1.7}$$

where we have introduced the notation

$$V_0(t,s) = \int_s^t B(\tau)U_0(\tau,s)\,d\tau, \tag{1.8}$$

$$V(t,s) = \int_s^t B(\tau)U(\tau,s)\,d\tau, \tag{1.9}$$

and the integral in (1.7) is to be understood as a Stieltjes integral.

For a given state $x \in X$ at time s, $B(t)U_0(t,s)x$ may be considered as the instantaneous output at time t of the unperturbed system (which is immediately fed back into the system). $V_0(t,s)x$ then stands for the *cumulative* output from time s up to time t. A formal manipulation shows that V_0 is connected to U_0 through the algebraic relation

$$V_0(t,r)x - V_0(s,r)x = V_0(t,s)U_0(s,r)x, \quad 0 \le r \le s \le t, \quad x \in X. \tag{1.10}$$

The identity (1.10) has a clear and useful interpretation. The difference on the left hand side gives the cumulative output from time s to time t of the unperturbed system, which had state x at time r, and this is alternatively obtained by letting the system evolve from time r to time s and then calculating the cumulative output of this new state from time s up to time t.

Notice that using (1.9) the variation-of-constants formula (1.6) can be rewritten as

$$U(t,s) = U_0(t,s) + \int_s^t U_0(t,\tau)V(d\tau,s), \tag{1.11}$$

which is an explicit formula for the perturbed evolutionary system U once V has been solved from (1.7).

We emphasize once more that our manipulations above have been entirely formal, that we have not defined the abstract Stieltjes integrals occurring in formulae (1.6), (1.7), and (1.11), and that the operators $A_0(t)$ and $B(t)$ need not be well-defined. The key point in our approach is to forget about these operators and define the perturbation problem as follows: Given an evolutionary system U_0 and a corresponding cumulative output V_0 (i.e., an operator family related to U_0 by (1.10)), solve the Stieltjes integral equation (1.7) and define U by the explicit formula (1.11) and finally prove that U is an evolutionary system and that V is a cumulative output for U.

We have formulated the problem above in forward terms since these are easier to interpret. As is well known (cf. Feller (1966)), the corresponding backward problem is often easier to solve and we shall therefore concentrate on the pre-dual problem of perturbing a backward evolutionary system by a so called step response. Results for the forward problem are then obtained by taking adjoints.

The perturbation problem mentioned above has been treated in the autonomous case by Diekmann, Gyllenberg and Thieme (1993a) and it is the purpose of this paper to extend those results to time-dependent problems.

In Section 2 we give rigorous definitions of backward and forward evolutionary systems and of step responses and cumulative outputs. We then develop the theory from a purely algebraic point of view assuming that a Stieltjes integral having all the usual properties has been defined. The abstract integral is defined and its basic properties are derived in Section 3. Section 4 is devoted to verifying that our assumptions are indeed satisfied in the case of strongly continuous backward evolutionary systems and in Section 5 we draw the appropriate conclusions for such systems and their adjoints. In Section 6 we investigate continuous dependence of solutions on the data. Our definition of the Stieltjes integral is in the spirit of Hönig (1975) using operator families of bounded semivariation. Many of the results in Sections 3 and 4 can be found in a slightly different guise in that book. For the sake of completeness and convenience of the reader we have included precise statements of these results. Full proofs can be found in Diekmann, Gyllenberg and Thieme (1993b). We would like to mention that Stieltjes integrals have been used in semigroup theory before to study the regularity of solutions to inhomogeneous Cauchy problems. Webb (1977) considers inhomogeneities of bounded variation, while Travis (1981) studies C_0 semigroups of bounded semivariation.

The perturbation problem can be solved if the step response of the unperturbed system satisfies a certain regularity condition, which can be hard to verify in applications. It is a very pleasant fact — proved in Section 7 — that this regularity condition is automatically satisfied for positive evolutionary systems perturbed by positive step responses. As nobody gets a negative number of children, this kind of positivity is a main feature of structured population models.

In Section 8 we apply the theory to structured population dynamics. Here it becomes clear that our approach is very similar to the branching process approach to population dynamics (cf. Jagers (1975,1989,1991,1992), Kimmel (1982,1983), Arino and Kimmel (1993)). In Section 9 we formulate a concrete size-structured population model in cumulative terms and point out the advantages of this formulation in comparison to the traditional PDE formulation.

2. Cumulative outputs, step responses, resolvents, and perturbation of evolutionary systems. Let $\rho \in \mathbf{R}$, $\tau \in (\rho, \infty]$ and let $\Delta_{\rho,\tau}$ denote the triangle

$$\Delta_{\rho,\tau} = \{(r,t);\ \rho \le r \le t < \tau\}.$$

If $\tau < \infty$, the closure of $\Delta_{\rho,\tau}$ is of course given by

$$\bar{\Delta}_{\rho,\tau} = \{(r,t);\ \rho \le r \le t \le \tau\}.$$

We also introduce

$$\Delta^*_{\tau,\rho} = \{(t,r);\ \rho \le r \le t < \tau\}$$

and

$$\bar{\Delta}^*_{\tau,\rho} = \{(t,r);\ \rho \le r \le t \le \tau\}.$$

A family $U = \{U(r,t)\}_{(r,t)\in\Delta_{\rho,\tau}}$, of bounded linear operators on the Banach space X is called a backward evolutionary system if

$$U(r,t) = U(r,s)U(s,t), \quad \rho \le r \le s \le t < \tau,$$
$$U(r,r) = I, \quad \rho \le r < \tau,$$

with I denoting the identity map on X. A family $U = \{U(t,r)\}_{(t,r)\in\Delta^*_{\tau,\rho}}$, of bounded linear operators on the Banach space X is called a forward evolutionary system if

$$U(t,r) = U(t,s)U(s,r), \quad \rho \le r \le s \le t < \tau,$$
$$U(r,r) = I, \quad \rho \le r < \tau.$$

At this point we do not specify any continuity properties of U. Notice that the dual U^* of a backward evolutionary system $U(r,t)$ defined by

$$U^*(t,r) = (U(r,t))^*$$

is a forward evolutionary system and vice versa. Notice the change in the order of the arguments.

Definition 2.1. A family $V = \{V(t,r)\}_{(t,r)\in\Delta^*_{\tau,\rho}}$ of bounded linear operators from X to a Banach space Y is called a *cumulative output* family for a forward evolutionary system U if

(i) $V(t,r) - V(s,r) = V(t,s)U(s,r), \quad \rho \le r \le s \le t < \tau.$

A family $V = \{V(r,t)\}_{(r,t)\in\Delta_{\rho,\tau}}$ of bounded linear operators from Y to X is called a *step response* for a backward evolutionary system U if

(i)′ $V(r,t) - V(r,s) = U(r,s)V(s,t), \quad \rho \le r \le s \le t < \tau.$

Setting $r = s = t$ in either (i) or (i)′ we see that necessarily

(ii) $V(r,r) = 0, \quad \rho \le r < \tau.$

If $Y = X$ we say that V is a cumulative output (step response) *on* X.

It is of course possible and meaningful to consider step responses of forward evolutionary systems and cumulative outputs of backward evolutionary systems. The defining relations are

$$V(t,r) - V(t,s) = U(t,s)V(s,r), \quad \rho \le r \le s \le t < \tau$$

for a step response V of a forward evolutionary system U and

$$V(r,t) - V(s,t) = V(r,s)U(s,t), \quad \rho \le r \le s \le t < \tau$$

for a cumulative output V of a backward evolutionary system U.

A step response has the interpretation suggested by the name: in the backward case $V(r,t)x$ is the state at time r when the final state at time t is zero and a constant input x is applied to the system in the interval (r,t) and in the forward case $V(t,r)x$ is the state at time t when the initial state at time r is zero and a constant input x is applied in (r,t).

In this paper we concentrate on the setting of Definition 2.1, because that gives what we need in the case of structured population problems.

The next elementary proposition states that cumulative output and step response are dual concepts.

Proposition 2.2. Let U be a backward evolutionary system on a Banach space X and let V be a step response from Y to X for U. Then $V^*(t,r) = (V(r,t))^*$ is a cumulative output from X^* to Y^* for U^*.

We are interested in the following *perturbation problem*: Given a backward evolutionary system U_0 and a step response V_0 for U_0, under what conditions does the Stieltjes integral equation

$$V(s,t) = V_0(s,t) + \int_s^t V(s,d\tau)V_0(\tau,t) \tag{2.1}$$

have a unique solution V? Is it true that U defined by

$$U(s,t) = U_0(s,t) + \int_s^t V(s,d\tau)U_0(\tau,t) \tag{2.2}$$

is a backward evolutionary system and that V is a step response for U? The problem of course subsumes the question of existence of the Stieltjes integrals in (2.1) and (2.2). We obviously also have a dual problem in terms of forward evolutionary systems and cumulative outputs.

To begin with we shall neglect the problem of defining Stieltjes integrals and concentrate on the actual perturbation problem. Equation (2.1) has the form of a *resolvent equation* and we shall therefore start by recalling from Gripenberg et al. (1990, Ch. 9.3) some purely algebraic facts about resolvent kernels.

Let $\mathcal{A}$ be an associative algebra with product $\star$. If $V_0 \in \mathcal{A}$ and $V \in \mathcal{A}$ satisfies the equation

$$V = V_0 + V_0 \star V = V_0 + V \star V_0, \tag{2.3}$$

then V is called the *resolvent kernel* of V_0. If it exists, the resolvent kernel is necessarily unique.

A *left module* over $\mathcal{A}$ is an Abelian group $\mathcal{M}$ such that the elements of $\mathcal{A}$ induce endomorphisms $f \to V \star f$ on $\mathcal{M}$ satisfying the following familiar laws:

$$V \star (f+g) = V \star f + V \star g, \quad V \in \mathcal{A}, \quad f, g \in \mathcal{M}, \tag{2.4}$$

$$(V+W) \star f = V \star f + W \star f, \quad V, W \in \mathcal{A}, \quad f \in \mathcal{M}, \tag{2.5}$$

$$(V \star W) \star f = V \star (W \star f), \quad V, W \in \mathcal{A}, \quad f \in \mathcal{M}. \tag{2.6}$$

Right modules are defined analogously, see Jacobson (1951) or any other textbook on algebra.

As is customary, we have denoted the product in the algebra and the operation inducing the endomorphisms by the same symbol $\star$. In our applications the $\star$ will always denote some sort of a Stieltjes product (see (2.14) below).

Proposition 2.3 (Gripenberg et al. (1990, Lemma 3.4, p. 233)). *Let $\mathcal{M}$ be a left module over $\mathcal{A}$ and let $U_0 \in \mathcal{M}$. If $V_0 \in \mathcal{A}$ has a resolvent $V \in \mathcal{A}$, then the equation*

$$U = U_0 + V_0 \star U \tag{2.7}$$

has a unique solution $U \in \mathcal{M}$. This solution is given by the variation-of-constants formula

$$U = U_0 + V \star U_0. \tag{2.8}$$

The following analogue for right modules is proved in exactly the same manner.

Proposition 2.4. *Let $\mathcal{M}$ be a right module over $\mathcal{A}$ and let $W_0 \in \mathcal{M}$. If $V_0 \in \mathcal{A}$ has a resolvent $V \in \mathcal{A}$, then the equation*

$$W = W_0 + W \star V_0$$

is uniquely solved by

$$W = W_0 + W_0 \star V\,.$$

In Section 3.2 we shall define a Stieltjes integral $\int_r^t V(d\tau)W(\tau) = \int_r^t d_\tau[V(\tau)]W(\tau)$ for $\mathcal{L}(X)$-valued functions with the following natural properties (at least under appropriate continuity assumptions):

$$\int_r^r V(d\tau)W(\tau) = 0, \tag{2.9}$$

$$\int_r^t V(d\tau)W(\tau) = \int_r^s V(d\tau)W(\tau) + \int_s^t V(d\tau)W(\tau), \quad r \le s \le t \tag{2.10}$$

$$\int_r^t V(d\tau)Z(\tau) = \int_r^t W(d\tau)Z(\tau), \tag{2.11}$$

whenever $W = V + C$ with $C \in \mathcal{L}(X)$ constant. For $C \in \mathcal{L}(X)$ the following equations also hold

$$\int_r^t d_\tau[CV(\tau)]W(\tau) = C\int_r^t V(d\tau)W(\tau), \tag{2.12}$$

$$\int_r^t ([V(d\tau)]W(\tau)C) = (\int_r^t V(d\tau)W(\tau))C. \tag{2.13}$$

Any object called an integral has to satisfy (2.10), whereas (2.9) and (2.11) are characteristic of Stieltjes integrals. As integrals are defined by limits of sums the properties (2.12) and (2.13) are natural as well. We shall also show that for a large class $\mathcal{A}_{\rho,\tau}$ of Volterra operator kernels, i.e., operator families defined on some triangle $\Delta_{\rho,\tau}$, the Stieltjes product

$$(V \star W)(s,t) := \int_s^t V(s, d\tau)W(\tau, t) \tag{2.14}$$

is well defined and makes $\mathcal{A}_{\rho',\tau'}$ into an associative algebra for all ρ', τ' satisfying $\rho \le \rho' \le \tau' \le \tau$.

Theorem 2.5. *Let $\mathcal{A}_{\rho,\tau}$ be an algebra of Volterra kernels on $\Delta_{\rho,\tau}$ with respect to the Stieltjes product* (2.14) *and let $\mathcal{M}_{\rho,\tau}$ be a set of Volterra kernels on $\Delta_{\rho,\tau}$ that forms a left module over $\mathcal{A}_{\rho,\tau}$ (also with respect to the Stieltjes product* (2.14)*). Let $U_0 \in \mathcal{M}_{\rho,\tau}$ be a backward evolutionary system and let $V_0 \in \mathcal{A}_{\rho,\tau}$ be a step response for U_0 having a resolvent $V \in \mathcal{A}_{\rho,\tau}$. Define U by*

$$U = U_0 + V \star U_0. \tag{2.15}$$

Then $U \in \mathcal{M}_{\rho,\tau}$, U is a backward evolutionary system and V is a step response for U on X. Moreover, U is the unique solution of the equation

$$U = U_0 + V_0 \star U. \tag{2.16}$$

Proof. For $r \le s \le t$ we have

$$\begin{aligned}
V(r,t) =& V_0(r,t) + \int_r^t V(r,d\rho)V_0(\rho,t)\\
=& V_0(r,s) + U_0(r,s)V_0(s,t) + \int_r^s V(r,d\rho)[V_0(\rho,s) + U_0(\rho,s)V_0(s,t)]\\
&+ \int_s^t d_\rho[V(r,\rho) - V(r,s)]V_0(\rho,t)\\
=& V_0(r,s) + \int_r^s V(r,d\rho)V_0(\rho,s) + \left[U_0(r,s) + \int_r^s V(r,d\rho)U_0(\rho,s)\right]V_0(s,t)\\
&+ \int_s^t d_\rho[V(r,\rho) - V(r,s)]V_0(\rho,t)\\
=& V(r,s) + U(r,s)V_0(s,t) + \int_s^t d_\rho[V(r,\rho) - V(r,s)]V_0(\rho,t),
\end{aligned} \tag{2.17}$$

where we have used (2.1), (2.15), the fact that V_0 is a step response for U_0, and the properties (2.9)–(2.11), (2.13) of the Stieltjes integral. Fix $(r,s) \in \Delta_{\rho,\tau}$. Setting

$$W(t) := V(r,t) - V(r,s), \qquad W_0(t) := U(r,s)V_0(s,t), \tag{2.18}$$

for $s \le t < \tau$ we can rewrite the identity (2.17) as

$$W(t) = W_0(t) + \int_s^t W(d\sigma)V_0(\sigma,t), \quad s \le t < \tau. \tag{2.19}$$

Clearly all functions on $[s,\tau)$ of the type $t \mapsto CZ(s,t)$ with $C \in \mathcal{L}(X)$ and $Z \in \mathcal{A}_{s,\tau}$ form a right module $\mathcal{M}$ over $\mathcal{A}_{s,\tau}$ with respect to

$$(W \star V)(t) = \int_s^t W(d\sigma)V(\sigma,t), \quad s \le t < \tau$$

for $W \in \mathcal{M}$, $V \in \mathcal{A}_{s,\tau}$. As W_0 belongs to $\mathcal{M}$ it follows from Proposition 2.4 that W is given by

$$W(t) = W_0(t) + \int_s^t W_0(d\rho)V(\rho,t), \quad s \le t < \tau. \tag{2.20}$$

Substituting (2.18) into (2.20) we find

$$\begin{aligned} V(r,t) - V(r,s) &= U(r,s)V_0(s,t) + \int_s^t d_\rho[U(r,s)V_0(s,\rho)]V(\rho,t) \\ &= U(r,s)V_0(s,t) + U(r,s)\int_s^t d_\rho[V_0(s,\rho)]V(\rho,t) = U(r,s)V(s,t), \end{aligned}$$

which shows that V and U are related by condition (i)′ of Definition 2.1.

Since $\mathcal{M}_{\rho,\tau}$ is a left module it is clear that U defined by (2.15) is also an element of $\mathcal{M}_{\rho,\tau}$ and, by Proposition 2.3, U is the unique solution of equation (2.16). To complete the proof we check that U is an evolutionary system. Since $U_0(r,r) = I$, it follows from (2.15) and (2.9) that $U(r,r) = I$. Let $r \le s \le t$. Then

$$\begin{aligned} U(r,t) &= U_0(r,t) + \int_r^t V(r,d\rho)U_0(\rho,t) \\ &= U_0(r,s)U_0(s,t) + \int_r^s V(r,d\rho)U_0(\rho,s)U_0(s,t) + \int_s^t d_\rho[V(r,\rho) - V(r,s)]U_0(\rho,t) \\ &= \left[U_0(r,s) + \int_r^s V(r,d\rho)U_0(\rho,s)\right]U_0(s,t) + \int_s^t d_\rho[U(r,s)V(s,\rho)]U_0(\rho,t) \\ &= U(r,s)\left[U_0(s,t) + \int_s^t V(s,d\rho)U_0(\rho,t)\right] = U(r,s)U(s,t), \end{aligned}$$

where we have again used the properties (2.9) -(2.13) of the Stieltjes integral. □

Theorem 2.5 shows that the perturbation problem is reduced to the question whether or not the step response of the unperturbed system has a resolvent kernel. The standard way to prove existence of a resolvent kernel is to actually construct it through the series expansion

$$V = \sum_{j=0}^{\infty} V_n, \qquad V_{n+1} = V_0 \star V_n. \tag{2.21}$$

The main problem is now to define a reasonable sense in which the series in (2.21) converges. In some applications this is very easy. In models of structured population dynamics $V_0^*(t,s)$ are the reproduction operators giving the expected number of children in the time interval $[s,t]$ as distributed with respect to their individual state at birth. $V_1^*(t,s)$ corresponds in the same manner to grand children, $V_2^*(t,s)$ to great grand children and so on (cf. Diekmann, Gyllenberg, Metz, and Thieme (1994) for more details). If one assumes that neonates cannot immediately produce offspring of their own, then for any fixed (t,s) at most finitely many of the operators $V_n^*(t,s)$ are non-zero and hence the sum in (2.21) contains at most finitely many terms for every fixed (s,t). Hence there is no convergence problem.

To treat the general case we shall in Section 4 equip an algebra of Volterra kernels with a locally convex vector topology that is also compatible with the Stieltjes product $\star$. We shall then show that if V_0 satisfies an additional regularity condition, then the series converges with respect to this topology and thus V_0 has a resolvent.

3. Stieltjes integration with operator-valued functions of bounded semi-variation. In this section we analyze the properties of the Stieltjes integral introduced by Hönig (1975) to justify the formal calculations of the preceding section. Most of the proofs are omitted and can be found in Diekmann, Gyllenberg and Thieme (1993b), which is a more detailed precursor of the present paper.

3.1. Operator-valued functions of bounded semi-variation. Let X, Y, Z be Banach spaces and let $V : [r, t] \to \mathcal{L}(Y, Z)$ be a family of bounded linear operators from Y to Z. We want to define Stieltjes integrals

$$\int_r^t V(ds)y(s)$$

for continuous functions $y : [r, t] \to Y$ and

$$\int_r^t V(ds)W(s)$$

for operator families $W : [r, t] \to \mathcal{L}(X, Y)$. In order to treat the strong operator topology and uniform operator topology cases simultaneously we use the notion of a bilinear triple (cf. Hönig, 1975).

By a *bilinear triple* $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ we understand a triple of Banach spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, with a bilinear continuous mapping $\mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, then $xy \in \mathcal{Z}$ denotes the value of the bilinear mapping. After equivalent renormalization we can assume that

$$\|xy\| \leq \|x\| \|y\|. \tag{3.1}$$

As pointed out above, the main examples we have in mind are $\mathcal{X} = \mathcal{L}(Y, Z)$, $\mathcal{Y} = Y$, $\mathcal{Z} = Z$ and $\mathcal{X} = \mathcal{L}(Y, Z)$, $\mathcal{Y} = \mathcal{L}(X, Y)$, $\mathcal{Z} = \mathcal{L}(X, Z)$ for given Banach spaces X, Y, Z. $\mathcal{L}(X, Y)$ denotes the bounded linear operators from X to Y with the uniform operator topology. The respective bilinear mappings are the application of an operator to an element and the composition of two operators. In these cases the inequality (3.1) holds by definition without any renormalization.

Given a bilinear triple it is possible to define a Stieltjes integral for step functions. A function $y : [r, t] \to \mathcal{Y}$ is called a *step function* (cf. Dieudonné (1969)) if there exists a partition $r = t_0 < \cdots < t_{n+1} = t$ such that y is constant on (t_j, t_{j+1}) for $j = 0, \ldots, n$. The space of step functions on $[r, t]$ with values in $\mathcal{Y}$ is denoted by $P([r, t], \mathcal{Y})$. Let $\bar{P}([r, t], \mathcal{Y})$ denote its closure under the supremum norm. Elements of $\bar{P}([r, t], \mathcal{Y})$ are called *regulated functions* (Dieudonné,1969). Notice that $\bar{P}([r, t], \mathcal{Y})$ contains the continuous functions on $[r, t]$ with values in $\mathcal{Y}$. Equivalent characterizations of regulated functions can be found in (Dieudonné, 1969, p. 145).

If x and y are functions defined on $[r, t]$ with values in $\mathcal{X}$ and $\mathcal{Y}$, respectively, y a step function, then we define the Stieltjes integral

$$\begin{aligned} &\int_r^t x(ds)y(s) = \sum_{j=0}^{n} \Big(x(t_{j+1}) - x(t_j)\Big) y(s_j), \quad r < t \\ &\int_r^r x(ds)y(s) = 0, \end{aligned} \tag{3.2}$$

where the partition $r = t_0 < \cdots < t_{n+1} = t$ has been chosen as above and $s_j \in (t_j, t_{j+1})$. The definition is independent of the choice of t_j, s_j.

We recall that a function $x : [r, t] \to \mathcal{X}$ into a Banach space $\mathcal{X}$ is of bounded variation if

$$\mathbf{v}_x^\bullet(r; t) := \sup \left\{ \sum_{j=0}^{n} \|x(t_{j+1}) - x(t_j)\| \right\} < \infty, \tag{3.3}$$

where the supremum is taken over all partitions $r = t_0 < \ldots < t_{n+1} = t$. The number $\mathbf{v}_x^\bullet(r;t)$ is called the variation of x on $[r,t]$. If x is of bounded variation, then it is certainly possible to extend the Stieltjes integral from the step functions to the continuous or even to the bounded Borel measurable functions y from $[r,t]$ to $\mathcal{Y}$ (see Section 3.3). Notice that the variation is additive, i.e.,

$$\mathbf{v}_x^\bullet(r;t) = \mathbf{v}_x^\bullet(r;s) + \mathbf{v}_x^\bullet(s;t), \quad r \leq s \leq t.$$

This leads to the estimate

$$\left\| \int_r^t x(ds)y(s) \right\| \leq \int_r^t \|y(s)\| \mathbf{v}_x^\bullet(r;ds) \leq \sup_{s\in[r;t]} \|y(s)\| v_x^\bullet(r;t), \quad y \in P([r,t],\mathcal{Y}), \tag{3.4}$$

with the second integral being a standard scalar Stieltjes integral.

If $\mathcal{X} = \mathcal{L}(Y,Z)$ for some Banach spaces Y, Z, the notion of bounded variation is over restrictive as we have pointed out in Example 5.1 in Diekmann, Gyllenberg and Thieme (1993a).

Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple. A function $x : [r,t] \to \mathcal{X}$ is said to be of *bounded $\mathcal{B}$-variation* on $[r,t]$ if

$$\mathbf{v}_x^{\mathcal{B}}(r;t) := \sup \left\{ \left\| \sum_{j=0}^{n} \big(x(t_{j+1}) - x(t_j)\big) y_j \right\| \right\} < \infty, \tag{3.5}$$

where the supremum is taken over all partitions

$$r = t_0 < \ldots < t_{n+1} = t \quad \text{and all} \quad y_j \in Y \quad \text{with} \quad \|y_j\| \leq 1.$$

The number $\mathbf{v}_x^{\mathcal{B}}(r;t)$ is called the *$\mathcal{B}$-variation* of x on $[r,t]$.

Notice that, in the definition of $\mathbf{v}_x^{\mathcal{B}}$, the partition $r = t_0 < \ldots < t_{n+1} = t$ can be replaced by $r \leq t_0 < \ldots < t_{n+1} \leq t$. This is possible by choosing some of the y_j equal to 0. After this remark one easily sees that

$$\mathbf{v}_x^{\mathcal{B}}(r;t) \leq \mathbf{v}_x^{\mathcal{B}}(r;s) + \mathbf{v}_x^{\mathcal{B}}(s;t), \quad r \leq s \leq t. \tag{3.6}$$

It follows that $\mathbf{v}_x^{\mathcal{B}}(r;t)$ is monotone non-increasing in r and monotone non-decreasing in t. The $\mathcal{B}$-variation fails to be additive and there is no analogue of formula (3.4). This is why integration with respect to functions of bounded $\mathcal{B}$-variation will be less satisfactory than with respect to functions of bounded variation.

A function $x : [\sigma,\tau) \to \mathcal{X}$ is said to be *locally of bounded $\mathcal{B}$-variation* (on $[\sigma,\tau)$) if it is of bounded $\mathcal{B}$-variation on every interval $[\sigma,t]$ with $\sigma \leq t < \tau$.

Let us connect this more general notion of bounded variation to the usual one.

Lemma 3.1. (a) *A function $x : [r,t] \to \mathcal{X}$ into a Banach space $\mathcal{X}$ is of bounded variation if and only if x is of bounded $\mathcal{B}^\bullet$-variation for the bilinear triple $\mathcal{B}^\bullet = (\mathcal{X}, \mathcal{X}^*, \mathbf{C})$ and*

$$\mathbf{v}_x^\bullet(r;t) = \mathbf{v}_x^{\mathcal{B}^\bullet}(r;t).$$

(b) *A function $x^* : [r,t] \to \mathcal{X}^*$ into the dual space of $\mathcal{X}$ is of bounded variation if and only if it is of bounded $\mathcal{B}^\diamond$-variation for $\mathcal{B}^\diamond = (\mathcal{X}^*, \mathcal{X}, \mathbf{C})$ and*

$$\mathbf{v}_{x^*}^\bullet(r;t) = \mathbf{v}_{x^*}^{\mathcal{B}^\diamond}(r;t).$$

As we have pointed out before, we shall mainly be concerned with two special cases, namely, for given Banach spaces X, Y, Z,

$$\mathcal{B}_1 = (\mathcal{L}(Y,Z), Y, Z) \quad \text{and} \quad \mathcal{B}_2 = (\mathcal{L}(Y,Z), \mathcal{L}(X,Y), \mathcal{L}(X,Z)),$$

with the natural bilinear mappings. Fortunately, for these two triples, the concepts of bounded $\mathcal{B}$-variation are the same. Moreover, one can always reduce a bilinear triple to a triple of the form of $\mathcal{B}_1$ which is quite convenient for some proofs.

Proposition 3.2 (Diekmann, Gyllenberg, and Thieme, 1993a, Proposition 2.1).
(a) *Let X, Y, Z be Banach spaces, $X \neq \{0\}$, $\mathcal{B}_1 = (\mathcal{L}(Y, Z), Y, Z)$ and let $\mathcal{B}_2 = (\mathcal{L}(Y, Z), \mathcal{L}(X, Y), \mathcal{L}(X, Z))$. Let V be a function from $[r, t]$ to $\mathcal{L}(Y, Z)$. Then V is of bounded $\mathcal{B}_1$-variation if and only if it is of bounded $\mathcal{B}_2$-variation and $\mathbf{v}_V^{\mathcal{B}_1}(r; t) = \mathbf{v}_V^{\mathcal{B}_2}(r; t)$.*

(b) *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple, $x : [r, t] \to \mathcal{X}$. Then $V(t)y := x(t)y$ defines a function $V : [r, t] \to \mathcal{L}(\mathcal{Y}, \mathcal{Z})$. We have that x is of bounded $\mathcal{B}$-variation if and only if V is of bounded $\mathcal{B}'$-variation for the bilinear triple $\mathcal{B}' = (\mathcal{L}(\mathcal{Y}, \mathcal{Z}), \mathcal{Y}, \mathcal{Z})$and $\mathbf{v}_x^{\mathcal{B}}(r; t) = \mathbf{v}_V^{\mathcal{B}'}(r; t)$. Moreover*

$$\int_r^t x(ds)y(s) = \int_r^t V(ds)y(s), \quad y \in P([r, t], \mathcal{Y}).$$

A function $V : [r, t] \to \mathcal{L}(Y, Z)$ which is of bounded $\mathcal{B}$-variation with respect to either (and hence both) of the bilinear triples $\mathcal{B}_1, \mathcal{B}_2$ of Proposition 3.2 (a) is said to be of *bounded semi-variation* on $[r, t]$. If $V : [0, \tau) \to \mathcal{L}(Y, Z)$ is of bounded semi-variation on every interval $[0, t]$, $0 \leq t < \tau$, then V is said to be *locally of bounded semi-variation.*

3.2. Stieltjes integration with operator-valued functions of bounded semi-variation. The following fundamental existence result which can be found in Hönig (1975), p. 24, explains the importance of the notion of bounded $\mathcal{B}$-variation. Similar results in somewhat more general settings can be found in Bartle (1956) Sections 1 and 2, and in Dinculeanu (1966), paragraphs 7 and 9.

Theorem 3.3. *Let $\mathcal{B} = (\mathcal{X}, Y, Z)$ be a bilinear triple and let $x : [r, t] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation. For $y \in \bar{P}([r, t], \mathcal{Y})$ (in particular if y is continuous) the Stieltjes integral*

$$\int_r^t x(ds)y(s)$$

exists. If y is a step function, the integral is given by (3.2). *If $r = t$, the integral is* 0. *The following estimate holds*

$$\| \int_r^t x(ds)y(s)\| \leq \mathbf{v}_x^{\mathcal{B}}(r, t) \sup_{r \leq s \leq t} \|y(s)\|. \tag{3.7}$$

Sometimes we shall also use the notation

$$\int_r^t d_s[x(s)]y(s) := \int_r^t x(ds)y(s).$$

We shall now proceed to show that the Stieltjes integral has all the properties (2.9)–(2.13) listed and used in Section 2. Property (2.9) is included in Theorem 3.3 and (2.11) follows from (3.7) since $\mathbf{v}_x^{\mathcal{B}}(r, t)$ is zero for constant functions x. Property (2.10) will be proved in Proposition 3.6 (b), (2.12) in Proposition 3.8, and (2.13) in Proposition 3.9.

We start by relating the Stieltjes integral to something more familiar through the following result on integration by parts:

Proposition 3.4. *Let $\mathcal{B} = (\mathcal{X}, Y, Z)$ be a bilinear triple and let $x : [r, t] \to \mathcal{X}$ be continuous and of bounded $\mathcal{B}$-variation. If $y : [r, t] \to \mathcal{Y}$ is continuously differentiable we have*

$$\int_r^t x(ds)y(s) = x(t)y(t) - x(r)y(r) - \int_r^t x(s)y'(s)ds$$

with the second integral being defined in the sense of Riemann.

In the following we need a couple of properties of the Stieltjes integral (cf. Hönig, 1975, Ch. I, Theorem 5.8).

Proposition 3.5. *Let $\mathcal{B} = (\mathcal{X}, Y, Z)$ be a bilinear triple and let $x : [r, t] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation. Let $x_k : [r, t] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation such that $\mathbf{v}^{\mathcal{B}}_{x_k}(r; t)$ is a bounded sequence and $x_k(s) \to x(s)$ for $k \to \infty$ pointwise in $s \in [r, t]$. Let $y, y_k \in \bar{P}([r, t], \mathcal{Y})$ and $y_k(s) \to y(s)$ for $k \to \infty$ uniformly in $s \in [r, t]$. Then*

$$\int_r^t x_k(ds)y_k(s) \to \int_r^t x(ds)y(s), \quad k \to \infty.$$

Our next proposition gives some results in the case of continuous functions y.

Proposition 3.6. *Let $\mathcal{B} = (\mathcal{X}, Y, Z)$ be a bilinear triple, let $x : [q, u] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation and let $y : [q, u] \to \mathcal{Y}$ be continuous.*

(a) *For $r \in [q, u]$ we have*

$$\left\| \int_q^u x(ds)y(s) - (x(u) - x(q))y(r) \right\| \le \mathbf{v}^{\mathcal{B}}_x(q; u) \sup_{q \le s \le u} \|y(s) - y(r)\|.$$

(b) *For $q \le r \le u$ one has*

$$\int_q^u x(ds)y(s) = \int_q^r x(ds)y(s) + \int_r^u x(ds)y(s).$$

(c) *If x, y are continuous on $[q, u]$, then $\int_r^t x(ds)y(s)$ is a continuous function of (r, t), $q \le r \le t \le u$.*

(d) *If x_j, y_j, $j \in J$, with J being some index set, are continuous functions from $[q, u]$ to $\mathcal{X}$, $\mathcal{Y}$ respectively and x_j is of bounded $\mathcal{B}$-variation on $[q, u]$ with $\mathbf{v}^{\mathcal{B}}_{x_j}(q; u)$ being bounded in $j \in J$ and if the families $\{x_j\}$ and $\{y_j\}$ are equicontinuous, then the family of functions $(r, t) \to \int_r^t x_j(ds)y_j(s)$ is equicontinuous.*

We shall also need the following Fubini type result, which is a special case of Hönig (1975, Ch. II, Theorem 1.1).

Proposition 3.7. *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple and let $x : [r, t] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation. Let $y : [r, t]^2 \to \mathcal{Y}$ be continuous. Then*

$$\int_r^t x(ds) \left(\int_r^t y(s, u)du \right) = \int_r^t \left(\int_r^t x(ds)y(s, u) \right) du.$$

Consider a bilinear triple $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and a bounded linear operator $B : \mathcal{Z} \to \mathcal{U}$ into a Banach space $\mathcal{U}$. Then the definition

$$(B^\sharp x)y = B(xy)$$

yields a bounded linear operator $B^\sharp : \mathcal{X} \to \mathcal{L}(\mathcal{Y}, \mathcal{U})$ with $\|B^\sharp\| \le \|B\|$. Let $\mathcal{B}^\sharp = (\mathcal{L}(\mathcal{Y}, \mathcal{U}), \mathcal{Y}, \mathcal{U})$ with the canonical pairing. In our two main examples where x is a bounded linear operator as well, $B^\sharp x = Bx$ is the composition of the two linear maps x and B.

Proposition 3.8. *Let $B \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ and let $x : [r, t] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation. Then $s \mapsto B^\sharp x(s)$ is of bounded $\mathcal{B}^\sharp$- variation,*

$$\mathbf{v}^{\mathcal{B}^\sharp}_{B^\sharp x(\cdot)}(r; t) \le \|B\| \mathbf{v}^{\mathcal{B}}_x(r; t) \quad \textit{and} \quad B \int_r^t x(ds)y(s) = \int_r^t d_s\big[B^\sharp x(s)\big]y(s).$$

Let us now explicitly turn to the case that $x = V$, $y = W$ are operator-valued functions, $V : [r, t] \to \mathcal{L}(Y, Z)$, $W : [r, t] \to \mathcal{L}(X, Y)$. We assume that V is of bounded semi-variation and that W is strongly continuous, i.e., $s \mapsto W(s)x$ is continuous for any $x \in X$. Then we can define a linear bounded operator $\int_r^t V(ds)W(s)$ from X to Z by

$$\left(\int_r^t V(ds)W(s)\right)x = \int_r^t V(ds)\big(W(s)x\big), \quad x \in X, \tag{3.8}$$

with the second integral being taken with respect to the bilinear triple $\mathcal{B}_1 = (\mathcal{L}(Y, Z), Y, Z)$. If W is operator-norm-continuous, we can also consider $\int_r^t V(ds)W(s)$ as an integral with respect to the bilinear triple $\mathcal{B}_2 = (\mathcal{L}(Y, Z), \mathcal{L}(X, Y), \mathcal{L}(X, Z))$. As one realizes from (3.2) and the approximation by step functions, the two integrals coincide. We note the following obvious consequence.

Proposition 3.9. *Let $V : [r, t] \to \mathcal{L}(Y, Z)$ be of bounded semi-variation and let $W : [r, t] \to \mathcal{L}(X, Y)$ be strongly continuous. Let $B : U \to X$ be a bounded linear operator. Then*

$$\left(\int_r^t V(ds)W(s)\right)B = \int_r^t V(ds)\big(W(s)B\big)$$

with the operator integrals being understood in the pointwise sense of (3.8). *If W is operator-norm-continuous, then they can also be understood with respect to the triple $\mathcal{B}_2$.*

3.3. Stieltjes integration with respect to functions of bounded variation. If $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a bilinear triple and $x : [r, t] \to \mathcal{X}$ is of bounded variation, then it is of bounded $\mathcal{B}$-variation. In particular the integral $\int_r^t x(ds)y(s)$ is defined for continuous $y : [r, t] \to \mathcal{Y}$. Moreover the estimate (3.4) still holds. The integral

$$\int_r^t \phi(s)\mathbf{v}_x^\bullet(r; ds)$$

is a non-negative continuous linear functional for continuous scalar-valued functions ϕ defined on $[r, t]$. Hence, by Riesz's representation theorem, there exists a non-negative finite regular Borel measure ξ on $[r, t]$ such that

$$\left\| \int_r^t x(ds)y(s) \right\| \le \int_r^t \|y(s)\| \xi(ds), \quad y \in C([r, t], \mathcal{Y}).$$

Hence the integral $\int_r^t x(ds)y(s)$ can be extended to functions $y \in L^1([r, t], \mathcal{Y}; \xi)$ with $L^1([r, t], \mathcal{Y}; \xi)$ denoting the completion of $C([r, t], \mathcal{Y})$ with respect to $\int_r^t \|y(s)\|\xi(ds)$. As ξ is regular, the integral is defined for bounded Borel measurable functions y. Finally a theorem of bounded pointwise convergence holds.

Proposition 3.10. *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple and $x : [r, t] \to \mathcal{X}$ be of bounded variation. Let $y_j : [r, t] \to \mathcal{Y}$ be a sequence of uniformly bounded Borel measurable functions such that $y_j(s) \to 0$, $j \to \infty$, pointwise in $s \in [r, t]$. Then*

$$\int_r^t x(ds)y_j(s) \to 0, \quad j \to \infty.$$

3.4. Duality. We specialize Proposition 3.8 to the case that $\mathcal{U} = \mathbf{C}$, hence $B = z^*$ for some $z^* \in \mathcal{Z}^*$. In this situation it is suggestive to write

$$(B^\sharp x)y = \langle xy, z^*\rangle =: \langle y, x^*z^*\rangle = (x^*z^*)y.$$

Indeed, the second equality defines a bounded linear operator $x^* : \mathcal{Z}^* \to \mathcal{Y}^*$ for any $x \in \mathcal{X}$. $\mathcal{B}^\sharp$ becomes $(\mathcal{Y}^*, \mathcal{Y}, \mathbf{C})$. Again, in our two main examples, where x is a bounded linear operator itself, x^* is the dual operator of x.

We have the following dual characterization of bounded $\mathcal{B}$-variation.

Proposition 3.11. (a) *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple. A function $x : [r, t] \to \mathcal{X}$ is of bounded $\mathcal{B}$-variation if and only if the function $s \mapsto x^*(s)z^*$ from $[r, t]$ into $\mathcal{Y}^*$ is of bounded variation for any $z^* \in \mathcal{Z}^*$.*
(b) *If x is of bounded $\mathcal{B}$-variation,*

$$\mathbf{v}_x(r; t) = \sup_{\|z^*\| \le 1} \mathbf{v}^\bullet_{x^*(\cdot)z^*}(r; t).$$

(c) *If x is of bounded $\mathcal{B}$-variation and $y : [r, t] \to \mathcal{Y}$ is continuous, then*

$$\left\langle \int_r^t x(ds)y(s), z^* \right\rangle = \int_r^t d_s\big[x^*(s)z^*\big]y(s), \quad \forall\, z^* \in Z^*.$$

Proof. Assume that x is of bounded $\mathcal{B}$-variation. Then, by Proposition 3.8, x^*z^* is of bounded $(\mathcal{Y}^*, \mathcal{Y}, \mathbf{C})$-variation, i.e., by Lemma 3.1 (b), of bounded variation. $\ge$ in (b) now follows from Lemma 3.1 (b) and Proposition 3.8.

Next we assume that $s \mapsto x^*(s)z^*$ from $[r, t]$ into $\mathcal{Y}^*$ is of bounded variation for any $z^* \in \mathcal{Z}^*$. Let us denote the set of $\mathcal{Y}$-valued step functions y with $\sup_{[r,t]} \|y\| \le 1$ by $P_1 = P_1([r, t], \mathcal{Y})$. For any $y \in P_1$, we define linear bounded operators $T_y : Z^* \to \mathbf{C}$ by

$$T_y(z^*) = \left\langle \int_r^t x(ds)y(s), z^* \right\rangle = \int_r^t (x^*(ds)z^*)y(s)$$

with the second integral being taken with respect to the bilinear triple $(\mathcal{Y}^*, \mathcal{Y}, \mathbf{C})$. Notice that

$$\left\| \int_r^t x(ds)y(s) \right\| = \|T_y\|. \tag{3.9}$$

From (3.4),

$$\sup_{y \in P_1} \|T_y(z^*)\| \le \mathbf{v}^\bullet_{x^*(\cdot)z^*}(r; t). \tag{3.10}$$

By the uniform boundedness theorem, there is some constant $c > 0$ such that $\|T_y\| < c$, $y \in P_1$. Let $r = t_0 < \cdots < t_{n+1}$ be a partition of $[r, t]$, $y_j \in \mathcal{Y}$, $\|y_j\| \le 1$. Define $y \in P_1$ by $y(s) = y_j$, $t_j < s < t_{j+1}$. Then

$$\left\| \sum_{j=0}^n \big(x(t_{j+1}) - x(t_j)\big)\, y_j \right\| = \left\| \int_r^t x(ds)y(s) \right\| = \|T_y\| < c.$$

Hence x is of bounded $\mathcal{B}$-variation. As for part (b), by (3.10),

$$\mathbf{v}_x(r; t) \le \sup_{\|z^*\| \le 1} \mathbf{v}^\bullet_{x^*(\cdot)z^*}(r; t) \le \mathbf{v}_x(r; t),$$

where the second inequality has already been proved at the beginning. Part (c) is a direct consequence of Proposition 3.8. □

In view of Proposition 3.5 one wonders whether Proposition 3.10 of bounded pointwise convergence (in y) would also hold if x is only of bounded semi-variation. We do not know about such a theorem. Even if one alternatively tries a more measure-theoretically oriented approach (see Bartle (1956)) a theorem of dominated convergence in measure seems to be the best one can achieve. A bounded convergence theorem holds in the weak topology, however.

Proposition 3.12. *Let y_j be a uniformly bounded sequence of continuous functions from $[r,t] \to Y$, and let $V : [r,t] \to \mathcal{L}(X,Y)$ be of bounded semi-variation. If $\|y_j(s)\| \to 0$, $j \to \infty$, pointwise for $s \in [r,t]$, then*

$$\left\langle \int_r^t V(ds)y_j(s), z^* \right\rangle \longrightarrow 0, \quad j \to \infty, \quad z^* \in Z^*.$$

If $V : [r,t] \to \mathcal{L}(Y,Z)$ is of bounded semi-variation, the functions $s \mapsto V^*(s)z^*$ are of bounded variation for any $z^* \in Z^*$ by Proposition 3.8. By Lemma 3.1 (a) this is equivalent to $s \mapsto V^*(s)z^*$ being of bounded $\mathcal{B}$-variation for $\mathcal{B} = (Y^*, Y^{**}, \mathbf{C})$. Hence we can define

$$\int_r^t d_s(V^*(s)z^*)y^{**}(s)$$

for any continuous $y^{**} : [r,t] \to Y^{**}$. Moreover

$$\left\| \int_r^t d_s(V^*(s)z^*)y^{**}(s) \right\| \le \sup_{[r,t]} \|y^{**}\| \, \|z^*\| \, \mathbf{v}_V(r;t).$$

Setting

$$\int_r^t y^{**}(s)V^*(ds)z^* = \int_r^t d_s(V^*(s)z^*)y^{**}(s)$$

we obtain an element

$$\int_r^t y^{**}(s)V^*(ds) \in Z^{**}.$$

Let J be the canonical embedding of Z into Z^{**}. Apparently, for continuous $y : [r,t] \to \mathcal{Y}$

$$\int_r^t V(ds)y(s) = \int_r^t Jy(s)V^*(ds).$$

In the following we identify Z with JZ and X with JX.

Let now $W^\times : [r,t] \to \mathcal{L}(Y^*, X^*)$ be a family of bounded linear operators such that $W^{\times *}(s)x$ is strongly continuous for any $x \in X$. Then we can define a w^*-integral $\int_r^t W^\times(s)V^*(ds) \in \mathcal{L}(Z^*, X^*)$ by setting

$$\left\langle x, \left(\int_r^t W^\times(s)V^*(ds) \right) z^* \right\rangle = \int_r^t d_s(V^*(s)z^*)W^{\times *}(s)x. \tag{3.11}$$

We have the estimate

$$\left\| \int_r^t W^\times(s)V^*(ds) \right\| \le \sup_{[r,t]} \|W^\times\| \mathbf{v}_V(r;t).$$

We conclude with the remark, that, for a strongly continuous operator family $U : [r, t] \to \mathcal{L}(X, Y)$, we have

$$\Big(\int_r^t V(ds)U(s)\Big)^* = \int_r^t U^*(s)V^*(ds). \tag{3.12}$$

4. The Stieltjes product and existence of a resolvent kernel. Let as before

$$\Delta_{\rho,\tau} = \{(r,t) : \rho \le r \le t < \tau\}, \quad \bar{\Delta}_{\rho,\sigma} = \{(r,t) : \rho \le r \le t < \sigma\},$$

where $\rho \in \mathbf{R}$, $\sigma \in (\rho, \infty)$, $\tau \in (\rho, \infty]$.

Definition 4.1. Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple. A Volterra kernel, that is a function $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{X}$, is called a $\mathcal{B}$-kernel if $V(r, \cdot)$ is of bounded $\mathcal{B}$-variation on $[r, \sigma]$ for any $r \in [\rho, \sigma]$ and

$$\mathbf{v}_V^{\mathcal{B}}(r; \sigma) := \mathbf{v}_{V(r,\cdot)}^{\mathcal{B}}(r, \sigma) \tag{4.1}$$

is a bounded function of $\rho \le r \le \sigma$.

The Volterra kernel $V : \Delta_{\rho,\tau} \to \mathcal{X}$ is called a local $\mathcal{B}$-kernel if it is a $\mathcal{B}$-kernel on $\bar{\Delta}_{\rho,\sigma}$ for any $\sigma \in (\rho, \tau)$.

It is convenient to introduce

$$\mathbf{V}_V^{\mathcal{B}}(r; t) = \sup_{r \le s \le t} \mathbf{v}_V^{\mathcal{B}}(s; t). \tag{4.2}$$

Then

$$\mathbf{V}_V^{\mathcal{B}}(r; t) = \sup \Big\| \sum_{j=0}^{n} \big(V(t_0, t_{j+1}) - V(t_0, t_j)\big) y_j \Big\|,$$

where the supremum is taken over all $r \le t_0 < \cdots < t_{n+1} \le t$ and all $y_j \in \mathcal{Y}$, $\|y_j\| \le 1$.

Let $\mathcal{B}_1 = (\mathcal{L}(X), X, X)$ and $\mathcal{B}_2 = (\mathcal{L}(X), \mathcal{L}(X), \mathcal{L}(X))$ for some Banach space X. It follows from Proposition 3.2 that a Volterra kernel is a $\mathcal{B}_1$-kernel if and only if it is a $\mathcal{B}_2$-kernel and that in this case

$$\mathbf{V}_V^{\mathcal{B}_1}(r; t) = \mathbf{V}_V^{\mathcal{B}_2}(r; t).$$

Definition 4.2. A Volterra kernel $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}(X)$ is called a Volterra-Stieltjes operator kernel if it is a $\mathcal{B}$-kernel with $\mathcal{B}$ either $\mathcal{B}_1$ or $\mathcal{B}_2$. We set

$$\mathbf{v}_V(r; t) = \mathbf{v}_V^{\mathcal{B}_1}(r; t).$$

A Volterra-Stieltjes operator kernel V is called *regular* if

$$\mathbf{v}_V(r; t) \to 0, \quad t - r \to 0. \tag{4.3}$$

Subsequently the word "operator" will usually be omitted.

Without an explicit statement to the contrary, topological concepts of Banach spaces like continuity will always refer to the norm topology. Thus, for instance, a continuous Volterra-Stieltjes operator kernel is a continuous mapping from $\bar{\Delta}_{\rho,\sigma}$ to $\mathcal{L}(X)$, equipped with the uniform operator topology.

As any continuous function $x : [0, t] \to X$ that is of bounded variation has the property that $\mathbf{v}_x^{\bullet}(0; s) \to 0, s \to 0$ (see Dunford and Schwartz (1958, III.5.16)), one might expect that a similar property is true for the $\mathcal{B}$-variation, or that any continuous Volterra-Stieltjes operator kernel is automatically regular. However, this conjecture is false as is shown by a counter example given in the Appendix.

Theorem 3.3 guarantees the existence of the Stieltjes product

$$(V \star x)(r,t) = \int_r^t V(r, ds)x(s,t), \quad 0 \le r < t < \tau, \tag{4.4}$$

satisfying $(V \star x)(r,r) = 0$ for a local Volterra-Stieltjes kernel V and a continuous function $x : \Delta_{\rho,\tau} \to X$. If $U : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ is continuous, then the Stieltjes product

$$(V \star U)(r,t) = \int_r^t V(r, ds)U(s,t), \quad 0 \le r < t < \tau, \tag{4.5}$$

makes sense in the uniform operator topology. If U is only strongly continuous, then the Stieltjes product (4.5) can still be defined pointwise:

$$(V \star U)(r,t)x = \int_r^t V(r, ds)U(s,t)x, \quad 0 \le r < t < \tau, \; x \in X. \tag{4.6}$$

The estimate (3.7) shows that $(V \star U)(r,t)$ is a bounded linear operator on X.

The next result also follows from the estimate in Theorem 3.3. Compare the proof of Proposition 2.4 in Diekmann, Gyllenberg and Thieme (1993a).

Proposition 4.3. *Let $\mathcal{B} = (\mathcal{X}, \mathcal{X}, \mathcal{X})$ be a bilinear triple and let $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{X}$ be a continuous $\mathcal{B}$-kernel, satisfying $V(s,s) = 0$ for all $\rho \le s \le \sigma$.*

(a) *Let $\rho \le r < \sigma$ and $y : [r,\sigma] \to \mathcal{X}$ be of bounded $\mathcal{B}$-variation. Then the function*

$$x(t) = \int_r^t y(ds)V(s,t), \quad r \le t \le \sigma,$$

is of bounded $\mathcal{B}$-variation on $[r,\sigma]$ and

$$\mathbf{v}_x^{\mathcal{B}}(r;t) \le \mathbf{v}_y^{\mathcal{B}}(r;t)\mathbf{V}_V^{\mathcal{B}}(r;t).$$

(b) *If $W : \bar{\Delta}_{\rho,\sigma} \to \mathcal{X}$ is a continuous $\mathcal{B}$-kernel, so is $W \star V$ and*

$$\mathbf{V}_{W\star V}^{\mathcal{B}}(r;t) \le \mathbf{V}_W^{\mathcal{B}}(r;t)\mathbf{V}_V^{\mathcal{B}}(r;t).$$

Proposition 4.4. *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple and let $W : \Delta_{\rho,\tau} \to \mathcal{X}$, $V : \Delta_{\rho,\tau} \to \mathcal{Y}$ be continuous functions. Assume that W is a local $\mathcal{B}$-kernel. Then $W \star V$ is a continuous function from $\Delta_{\rho,\tau}$ to $\mathcal{Z}$ that can continuously be extended to $[\rho,\tau)^2$ by setting it 0 on $[\rho,\tau)^2 \setminus \Delta_{\rho,\tau}$.*

Proof. Consider sequences r_j, t_j, $j \in \mathbf{N}$, such that $(r_j, t_j) \in \Delta_{\rho,\tau}$ and $r_j \to r, t_j \to t$, $j \to \infty$, $(r,t) \in \Delta_{\rho,\tau}$, $r < t$. Set $x_j(s) = W(r_j, s)$ if $s \ge r_j$ and $x_j(s) = 0$ otherwise, $y_j(s) = V(s, t_j)$ if $s \le t_j$ and $y_j(s) = 0$ otherwise, $x(s) = W(r,s)$, $y(s) = V(s,t)$. Then

$$\begin{aligned} &\|(W \star V)(r_j,t_j) - (W \star V)(r,t)\| \\ \le &\left\| \int_{r_j}^{t_j} x_j(ds)y_j(s) - \int_r^t x_j(ds)y_j(s) \right\| + \left\| \int_r^t x_j(ds)y_j(s) - \int_r^t x(ds)y(s) \right\|. \end{aligned}$$

The first term converges to 0 for $j \to \infty$ by Proposition 3.6 (d), because $\mathbf{v}_{x_j}^{\mathcal{B}}(\rho;\sigma)$ is bounded for some $\sigma \in (t,\tau)$ and the continuity of $y_j(s)$ in s is uniform in j. The second term converges to 0 for $j \to \infty$ by Proposition 3.5, because $y_j(s) \to y(s)$, $j \to \infty$ uniformly in $s \in [r,t]$. If $r = t$, we have $(W \star V)(r_j,t_j) \to 0 = (W \star V)(r,r)$ from Proposition 3.6 (a). □

The next proposition on the associativity of the Stieltjes products could be formulated in greater generality, but we prefer to state it only in terms of operator families.

Proposition 4.5. *Let $V : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ be a continuous local Volterra-Stieltjes kernel satisfying $V(r,r) = 0$, $\rho \le r < \tau$. Then the following statements hold:*

(a) *Let $\rho \le r \le t < \tau$ be fixed and $f : [r,t] \to X$ be continuous and $W : [r,t] \to \mathcal{L}(X,Y)$ be of bounded semi-variation. Then*

$$\int_r^t W(ds)\left(\int_s^t V(s,du)f(u)\right) = \int_r^t d_u\left[\int_r^u W(ds)V(s,u)\right]f(u). \tag{4.7}$$

(b) *If $U(r,t)$ is a strongly continuous family of bounded linear operators and V, W are continuous local Volterra-Stieltjes kernels from $\Delta_{\rho,\tau}$ to $\mathcal{L}(X)$ vanishing on the diagonal, then $W \star (V \star U) = (W \star V) \star U$.*

Proof. (a) First of all we notice that both double Stieltjes-integrals in (4.7) make sense. The same proof as for Proposition 4.4 shows that $\int_s^t V(s,du)f(u)$ is continuous and so the first double Stieltjes-integral is defined. By Proposition 4.3 the function $x(u) = \int_s^u W(ds)V(s,u)$ is locally of bounded semi-variation such that the second double Stieltjes integral is defined as well. We can assume that $f(s)$ is continuously differentiable in s. Otherwise we approximate f by continuously differentiable functions and the result follows from the estimates in Theorem 3.3 and Proposition 4.3. Integrating by parts (Proposition 3.4) we obtain

$$\int_r^t W(ds)\left(\int_s^t V(s,du)f(u)\right) = -\int_r^t W(ds)\left(\int_s^t V(s,u)f'(u)du - V(s,t)f(t)\right).$$

As $V(s,s) = 0$ we can extend V continuously by $V(s,u) = 0$ for $u \le s$ and so we can continue the identity by writing

$$= -\int_r^t W(ds)\left(\int_r^t V(s,u)f'(u)du - V(s,t)f(t)\right).$$

By Proposition 3.7 we can interchange the integration and continue our equation by

$$\begin{aligned} &= -\int_r^t \left(\int_r^t W(ds)V(s,u)\right) f'(u)du + \int_r^t W(ds)V(s,t)f(t) \\ &= -\int_r^t \left(\int_r^u W(ds)V(s,u)\right) f'(u)du + \int_r^t W(ds)V(s,t)f(t). \end{aligned}$$

Another integration by parts (Proposition 3.4) proves our claim.

The assertion in b) immediately follows from a) by fixing r and setting $f(s) = U(r,s)x$.

We collect our results so far in the following theorem.

Theorem 4.6. *Let X be a Banach space, $\rho < \sigma < \tau$. The set $\mathcal{A}_{\rho,\sigma}$ of all continuous Volterra-Stieltjes kernels $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}(X)$ satisfying $V(r,r) = 0$, $\rho \le r \le \sigma$, is a Banach algebra with Stieltjes product as multiplication and $\mathbf{V}_V(\rho;\sigma)$ as norm. The Abelian group (with respect to ordinary addition of operators) of all strongly continuous Volterra kernels $\bar{\Delta}_{\rho,\sigma} \to \mathcal{L}(X)$ is a left module over $\mathcal{A}_{\rho,\sigma}$.*

Theorem 4.6 implies a preliminary existence result for resolvent kernels. It is well known (Rudin, 1973, Theorem 10.7) that an element V_0 of a Banach algebra with norm less than one has a resolvent given by the Neumann series

$$V = \sum_{j=0}^{\infty} V_n, \quad V_{n+1} = V_0 \star V_n.$$

So every continuous Volterra-Stieltjes operator kernel $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}(X)$ vanishing on the diagonal and having semi-variation $\mathbf{V}_V(\rho,\sigma)$ less than one has a unique resolvent. In most applications this is too weak a result and we therefore proceed to prove existence of resolvents of regular elements of a larger algebra.

Let $\mathcal{A}$ denote the set of all continuous Volterra-Stieltjes operator kernels $V : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ that satisfy $V(r,r) = 0$ for all $r \in [\rho,\tau)$. In other words, $V \in \mathcal{A}$ if and only if the restriction of V to $\bar{\Delta}_{\rho,\sigma}$ belongs to $\mathcal{A}_{\rho,\sigma}$ for all $\sigma \in (\rho,\tau)$. $\mathcal{A}$ is obviously an algebra.

Define for all $V \in \mathcal{A}$ and all $\lambda \geq 0$

$$\|V\|_{\lambda,\sigma} = \sup \left\| \sum_{j=0}^{n} \big(V(t_0,t_{j+1}) - V(t_0,t_j)\big) e^{\lambda(t_0-t_j)} x_j \right\|, \tag{4.8}$$

where the supremum is taken over all partitions $\rho \leq t_0 < \cdots < t_{n+1} \leq \sigma$ and all elements $x_1, \ldots, x_{n+1} \in X$ with $\|x_j\| \leq 1$. For each $t \geq \rho, \lambda \geq 0$, the mapping $V \mapsto \|V\|_{\lambda,t}$ is a norm on $\mathcal{A}_{\rho,\sigma}$. The mapping $\sigma \mapsto \|V\|_{\lambda,\sigma}$ is monotone, non-decreasing, non-negative, and bounded on bounded intervals. For each σ, the norms $\|\cdot\|_{\lambda,\sigma}$ are equivalent to the norm $\|\cdot\|_{0,\sigma} = \mathbf{V}_V(\rho;\sigma)$, the norm introduced in Theorem 4.6. Notice that

$$\|V(r,t)\| \leq e^{\lambda(t-r)} \|V\|_{\lambda,\sigma}, \qquad \rho \leq r \leq t \leq \sigma$$

such that the convergence in $\|\cdot\|_{\lambda,\sigma}$ implies convergence in the operator norm uniformly on $\bar{\Delta}_{\rho,\sigma}$.

Lemma 4.7. (a) *If $V \in \mathcal{A}_\sigma$, and $\epsilon > 0$, then*

$$\|V\|_{\lambda,\sigma} \leq \sup_{0 \leq r \leq \sigma-\epsilon} \mathrm{v}_V(r; r+\epsilon) + e^{-\lambda\epsilon} \|V\|_{0,\sigma}.$$

(b) *If W and V belong to $\mathcal{A}_{\rho,\sigma}$ then*

$$\|(W \star V)\|_{\lambda,\sigma} \leq \|W\|_{\lambda,\sigma} \, \|V\|_{\lambda,\sigma}. \tag{4.9}$$

Proof (a) The statement is easily seen by introducing new partition points with elements $x_j = 0$ if needed.

(b) Let $0 \leq t_0 < \cdots t_{n+1} \leq \sigma,\ x_j \in X, \|x_j\| \leq 1$. Then

$$\begin{aligned}
&\left\| \sum_{j=1}^{n} \big((W \star V)(t_0,t_{j+1}) - (W \star V)(t_0,t_j)\big) e^{\lambda(t_0-t_j)} x_j \right\| \\
&= \left\| \sum_{j=1}^{n} \Big(\int_{t_0}^{t_{j+1}} W(t_0,ds) V(s,t_{j+1}) - \int_{t_0}^{t_j} W(t_0,ds) V(s,t_j) \Big) e^{\lambda(t_0-t_j)} x_j \right\|.
\end{aligned}$$

As $V(s,s) = 0$ we can extend V continuously by setting $V(s,u) = 0, s \geq u$ and obtain

$$\begin{aligned}
&\left\| \sum_{j=1}^{n} \big((W \star V)(t_0,t_{j+1}) - (W \star V)(t_0,t_j)\big) e^{\lambda(t_0-t_j)} x_j \right\| \\
&= \left\| \int_{t_0}^{\sigma} W(t_0,ds) \sum_{j=1}^{n} \big(V(s,t_{j+1}) - V(s,t_j)\big) e^{\lambda(t_0-t_j)} x_j \right\| = \left\| \int_{t_0}^{\sigma} W(t_0,ds) e^{\lambda(t_0-s)} f_\lambda(s) \right\|
\end{aligned}$$

with

$$f_\lambda(s) = \sum_{j=1}^{n} \Big(V(s, t_{j+1}) - V(s, t_j) \Big) e^{\lambda(s-t_j)} x_j. \tag{4.10}$$

$$\int_{t_0}^{\sigma} W(t_0, ds) e^{\lambda(t_0-s)} f_\lambda(s)$$

can be approximated by sums

$$\sum_{j=0}^{n} \Big(W(t_0, s_{j+1}) - W(t_0, s_j) \Big) e^{\lambda(t_0-s_j)} f_\lambda(s_j).$$

Hence its norm can be estimated from above by $\|W\|_{\lambda,\sigma} \sup_{0\le s\le\sigma} \|f_\lambda(s)\|$. By (4.10) we have

$$\|f_\lambda(s)\| \le \|V\|_{\lambda,\sigma}.$$

This implies the assertion.

Theorem 4.8. (a) *The set $\mathcal{A}$ of all continuous Volterra-Stieltjes operator kernels $V : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ vanishing on the diagonal is an algebra.*

(b) *For every fixed $r \in [\rho, \tau)$ the operation*

$$(V \star f)(t) := \int_r^t V(r, ds) f(s), \quad V \in \mathcal{A}, f \in C([r, \tau); X)$$

makes the Abelian group $C([r, \tau); X)$ of continuous functions into a left module over $\mathcal{A}$.

(c) *The Abelian group of all strongly continuous Volterra kernels $V : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ is a left module over $\mathcal{A}$.*

(d) *For every fixed $r \in [\rho, \tau)$ the operation*

$$(W \star V)(t) := \int_r^t W(ds) V(s, t), \quad V \in \mathcal{A}$$

makes the Abelian group of all functions $W : [\rho, \tau) \to \mathcal{L}(X, Y)$ that are locally of bounded semi-variation into a right module over $\mathcal{A}$.

(e) *For each $\lambda \ge 0$ the family $\{\|\cdot\|_{\lambda,\sigma}\}_{\sigma\in(\rho,\tau)}$ of norms defines a locally convex topology on the algebra $\mathcal{A}$ which is compatible with the algebraic structure. These topologies are equivalent for all $\lambda \ge 0$.*

(f) *Every regular $V_0 \in \mathcal{A}$ has a unique regular resolvent kernel $V \in \mathcal{A}$. V is given by the series*

$$V = \sum_{j=0}^{\infty} V_n, \qquad V_{n+1} = V_0 \star V_n. \tag{4.11}$$

which converges in $\mathcal{A}$.

Proof. (a) Obvious.

(b) By Theorem 3.3 $(V \star f)(t)$ is well defined and the same proof as for Proposition 4.4 shows that it is continuous in t (this was already observed in the proof of Proposition 4.5). Thus $f \to V \star f$ is an endomorphism of the Abelian group $C([r, \tau); X)$ and Proposition 4.5 (a) shows that $C([r, \tau); X)$ is a left module over $\mathcal{A}$.

(c) Obvious.

(d) This also follows from Proposition 4.5 in a similar manner as (b).

(e) By a standard result in functional analysis (Rudin,1973, Theorem 1.37) the separating family of (semi) norms induces a locally convex topology on $\mathcal{A}$ compatible with the vector space structure and Lemma 4.7 (b) shows that it is also compatible with the Stieltjes product. We have already noticed that the norms $\|\cdot\|_{\lambda,\sigma}$ are equivalent for all $\lambda \geq 0$ and hence the induced topologies are equivalent.

(f) If $V_0 \in \mathcal{A}$ is regular, it follows from Lemma 4.7 (a) that for fixed $\sigma \in (\rho, \tau)$ one can choose $\lambda \geq 0$ so large that $\|V_0\|_{\lambda,\sigma} < 1$. It follows that the series in (4.11) can be majorized in this norm by a converging geometric series and hence it converges (for details, see Diekmann, Gyllenberg and Thieme (1993a, Theorem 2.7).

We now apply Propositions 2.3 and 2.4 to the present situation and obtain the following theorem.

Theorem 4.9. *Let V_0 be a regular Volterra-Stieltjes kernel. Let V be the resolvent kernel of V_0. Let $\rho \leq r < \tau$ be fixed.*

(a) *Let f be a continuous function from $[r, \tau)$ to X. Then the Stieltjes integral equation*

$$v(t) = \int_r^t V_0(r, ds)v(s) + f(t), \quad r \leq t < \tau,$$

is uniquely solved by

$$v(t) = \int_r^t V(r, ds)f(s) + f(t), \quad r \leq t < \tau.$$

(b) *Let $U_0 : [r, \tau) \to \mathcal{L}(X)$ be a strongly continuous family of bounded linear operators on X. Then the operator Stieltjes integral equation*

$$U(t) = U_0(t) + \int_r^t V_0(r, ds)U(s), \quad r \leq t < \tau$$

is uniquely solved by

$$U(t) = U_0(t) + \int_r^t V(r, ds)U_0(s), \quad r \leq t < \tau.$$

(c) *Let $W_0 : [r, \tau) \to \mathcal{L}(X, Y)$ be locally of bounded semi-variation. Then the Stieltjes integral equation*

$$W(t) = W_0(t) + \int_r^t W(ds)V_0(s, t), \quad r \leq t < \tau,$$

is uniquely solved by

$$W(t) = W_0(t) + \int_r^t W_0(ds)V(s, t).$$

Remark 4.10. Translated into our framework, Hönig (1975, chapter 4, §1) shows existence and uniqueness of a Volterra-Stieltjes kernel

$$R(r, t) = I + \int_r^t V_0(r, ds)R(s, t) = I + \int_r^t R(r, ds)V_0(s, t), \quad \rho \leq r \leq t < \tau,$$

where I is the identity operator on X. Hönig calls R the resolvent kernel of V_0 and he uses R instead of V to solve the Stieltjes integral equations in Theorem 4.9. These two concepts of resolvent kernels can be transformed into each other, because one easily checks the following relations: If $V_0(r,r)=0$, $\rho \le r < \tau$, then

$$V(r,t) = \int_r^t V_0(r,ds)R(s,t) = \int_r^t R(r,ds)V_0(s,t),$$

and, from Theorem 4.9 (b), (c), $R(r,t) = I + V(r,t)$. Existence and uniqueness of both V and R as well as their relation were shown by Kimmel (1982) in the case that X is a finite-dimensional euclidean vector space. Kimmel (1982, 1983) also explains how they can be interpreted as certain moments in age- and time-dependent branching processes. We prefer V over R in our context because of its interpretation as step response of the perturbed evolutionary system U (see Sections 2 and 5). The resolvent kernel V will be a paramount tool for generalizing our theory to nonlinear perturbations.

5. Perturbation of strongly continuous and dual evolutionary systems.

Definition 5.1. A step response for a backward evolutionary system is called *regular* if it is a regular continuous local Volterra-Stieltjes operator kernel vanishing on the diagonal.

Theorem 5.2. *Let $U_0 = \{U_0(r,t)\}_{(r,t)\in\Delta_{\rho,\tau}}$ be a strongly continuous backward evolutionary system on a Banach space X and let $V_0 = \{V_0(r,t)\}_{(r,t)\in\Delta_{\rho,\tau}}$ be a regular step response for U_0 on X. Let V be the unique solution of the Stieltjes integral equation*

$$V(r,t) = V_0(r,t) + \int_r^t V(r,d\sigma)V_0(\sigma,t), \quad \rho \le r \le t \le \tau \tag{5.1}$$

and let U be defined by

$$U(r,t) = U_0(r,t) + \int_r^t V(r,d\rho)U_0(\rho,t), \quad \rho \le r \le t \le \tau. \tag{5.2}$$

Then U is a strongly continuous backward evolutionary system and V is a regular step response for U on X. Moreover,

$$U(r,t) = U_0(r,t) + \int_r^t V_0(r,d\sigma)U(\sigma,t), \quad \rho \le r \le t \le \tau. \tag{5.3}$$

Finally, V^* is a cumulative output for U^* and U^* is given by the variation of constant formula

$$U^*(t,r) = U_0^*(t,r) + \int_r^t U_0^*(t,s)V^*(ds,r), \quad \rho \le r \le t \le \tau. \tag{5.4}$$

The Stieltjes integral in (5.4) is in the weak*-sense, see Section 3.4, in particular formula (3.11)

Proof. By definition, V_0 is a regular element of the algebra $\mathcal{A}$ of Theorem 4.8. By that theorem V_0 has a unique (regular) resolvent kernel $V \in \mathcal{A}$. Since U_0 belongs to the left module of strongly continuous operator families, Theorem 2.5 implies that U is the unique solution of (5.3). That V^* is a cumulative output follows from Proposition 2.2 and finally (5.4) follows by taking adjoints of equation (5.2).

In the situation of Theorem 5.2 we say that the evolutionary system U and its step response V are obtained from the evolutionary system U_0 via *perturbation* by the step response V_0. Similarly we say that the dual system U^* and its cumulative output V^* are obtained via perturbation of the dual system U_0^* by the cumulative output V_0^*.

An immediate consequence of Theorem 5.2 is that for any $x \in X$ and $t \in (\rho, \tau]$, $u(r) := U(r,t)x$ is the unique solution of the backward equation

$$u(r) = U_0(r,t)x + \int_r^t V_0(r, d\sigma)u(\sigma).$$

Without additional assumptions a similar interpretation of $u^*(t) := U^*(t,r)x^*$, $x^* \in X^*$ as the solution of a forward equation is not valid. The next theorem tells us what kind of equation is satisfied by u^*.

Theorem 5.3. *Under the assumptions of* Theorem 5.2 *let* $r \in [\rho, \tau)$, $x^* \in X^*$ *and define* $u^*(t) := U^*(t,r)x^*$, $v^*(t) := V^*(t,r)x^*$. *Then* (u^*, v^*) *is the unique solution of the system*

$$u^*(t) = U_0^*(t,r)x^* + \int_r^t U_0^*(t,s)v^*(ds), \quad r \le t < \tau, \tag{5.5}$$

$$v^*(t) = V_0^*(t,r)x^* + \int_r^t v^*(ds)V_0(s,t), \quad r \le t < \tau. \tag{5.6}$$

$v^* : [r,\tau) \to X^*$ *is locally of bounded variation and (norm) continuous and* $u^* : [r,\tau) \to X^*$ *is weakly* continuous.*

Proof. Since $t \to V^*(t,r)x^*$ is locally of bounded semi-variation considered as a function from $[r,\tau)$ to $X^* = \mathcal{L}(X, \mathbf{C})$, we can apply Theorem 4.9 (c) to express the unique solution of (5.6) as

$$\begin{aligned} v^*(t) &= V_0^*(t,r)x^* + \int_r^t d_s[V_0^*(s,r)x^*]V(s,t) = V_0^*(t,r)x^* + \int_r^t d_s[x^*V_0(r,s)]V(s,t) \\ &= x^*\left(V_0(r,t) + \int_r^t V_0(r,ds)V(s,t)\right) = x^*V(r,t) = V^*(t,r)x^*. \end{aligned} \tag{5.7}$$

In the third equality in (5.7) we have used Proposition 3.8.

By Lemma 3.1 (b), v^* is in fact locally of bounded variation. Since V is continuous from $\Delta_{\rho,\tau}$ to $\mathcal{L}(X)$ equipped with the uniform operator topology and since the norm is preserved when taking adjoints, V^* is continuous from $\Delta^*_{\tau,\rho}$ to $\mathcal{L}(X^*)$ and hence v^* is continuous to X^* with its norm topology.

Since (5.5) is now an explicit formula there is no uniqueness problem for u^* and we only have to check the validity of the formula. To this end, notice that

$$\begin{aligned} \langle x, u^*(t)\rangle &= \langle U(r,t)x, x^*\rangle = \left\langle U_0(r,t)x + \int_r^t V(r,ds)U_0(s,t)x, x^*\right\rangle \\ &= \langle x, U_0^*(t,r)x^*\rangle + \int_r^t d_s[x^*V(r,s)]U_0(s,t)x \\ &= \langle x, U_0^*(t,r)x^*\rangle + \int_r^t v^*(ds)U_0(s,t)x, \end{aligned} \tag{5.8}$$

where the last Stieltjes integral has been taken with respect to the bilinear triple $(X^*, X, \mathbf{C})$. Since

$$\left\langle x, \int_r^t U_0^*(t,s)v^*(ds)\right\rangle = \int_r^t v^*(ds)U_0(s,t)x$$

we obtain

$$u^*(t) = U_0^*(t,r)x^* + \int_r^t U_0^*(t,r)v^*(ds),$$

that is, (5.5) holds. The weak*-continuity of u^* follows from (5.8) since U_0^* is weakly*-continuous.

6. Continuous dependence of solutions. The perturbation of strongly continuous backward evolutionary systems by step responses is surprisingly robust with respect to small changes of both the evolutionary system and the step response. This is important in view of applications where data can only be collected with certain errors.

We consider a sequence of strongly continuous backward evolutionary systems $U_j^\circ : \bar{\Delta}_{\rho,\sigma} \to X$ which converges in the strong operator topology uniformly on $\bar{\Delta}_{\rho,\sigma}$. Then the limit U° is also a strongly continuous backward evolutionary system. We have for any $x \in X$,

$$U_j^\circ(s,t)x \to U^\circ(s,t)x, \quad j \to \infty \text{ uniformly on } \bar{\Delta}_{\rho,\sigma}. \tag{6.1}$$

Further, there are regular step responses $V^\circ, V_j^\circ : \bar{\Delta}_{\rho,\sigma} \to X$ associated with U°, U_j° respectively with the following properties:

$$\|V_j^\circ(r,t) - V^\circ(r,t)\| \to 0, \quad j \to \infty, \text{ uniformly on } \bar{\Delta}_{\rho,\sigma}. \tag{6.2}$$

Further, we assume that the semi-variation of V_j° is bounded:

$$\sup_j \mathbf{V}_{V_j^\circ}(\rho;\sigma) < \infty. \tag{6.3}$$

Finally, we assume that their regularity is uniform in j:

$$\mathbf{v}_{V_j^\circ}(r;t) \to 0 \text{ for } t-r \searrow 0,\ \rho \le r < t \le \sigma, \text{ uniformly in } j \in \mathbf{N}. \tag{6.4}$$

Theorem 6.1. *Consider the setting described above. Let V_j, V be the resolvent kernel associated with V_j°, V° respectively and U_j, U be the evolutionary systems obtained from U_j°, U° via perturbation by the regular step responses V_j°, V°. Then the relations* (6.1) – (6.4) *are preserved under perturbation, i.e., U_j, U satisfy* (6.1) *and V_j, V satisfy* (6.2)– 6.4).

The surprising part of this theorem consists in the fact that we do not need the convergence of V_j° to V° in semi-variation (cf. Gripenberg et al., 1990, Lemma 9.3.11). The drawback of not assuming convergence in semi-variation consists in not obtaining any estimates for the speed of convergence. This problem will be addressed when we turn to quasilinear problems in future work.

The rest of this section is devoted to an outline of the proof of Theorem 6.1. We first notice that, by (6.3) and (6.4), we can choose a fixed $\lambda > 0$ such that the λ-norms defined in (4.8) satisfy

$$\|V_j^\circ\|_{\lambda,\sigma} \le 1/2 \quad \forall j \in \mathbf{N}.$$

This implies that the convergence of the series in (4.11) with respect to the λ-norm is uniform in j. In particular $\|V_j\|_{\lambda,\sigma} \leq 1$ which implies, by the equivalence to the original norm, that (6.3) holds for V_j. Moreover, by (2.3) and Theorem 4.9 (a)

$$\mathbf{v}_{V_j}(r,t) \leq \mathbf{v}_{V_j^\circ}(r,t)(1+\mathbf{V}_{V_j}(0,\sigma)).$$

Hence (6.4) is inherited by V_j from V_j°.

In order to show (6.2) for V_j, V we recall the remark following formula (4.8). It implies that the series in (4.11) converges in the operator norm, uniformly on $\bar{\Delta}_{\rho,\sigma}$ and uniformly in j. (6.2) now follows for V_j, V from (4.11) and from

$$\|V_{n,j}(r,t) - V_n(r,t)\| \to 0, \quad j \to \infty, \text{ uniformly for } 0 \leq r \leq t \leq \sigma$$

which, in turn, follows from the following lemma (with $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \mathcal{L}(X)$) by induction.

Lemma 6.2. *Let $\mathcal{B} = (\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a bilinear triple and W, $W_j : \bar{\Delta}_{\rho,\sigma} \to \mathcal{X}$ be continuous Stieltjes kernels, F, $F_j : \bar{\Delta}_{\rho,\sigma} \to \mathcal{Y}$ be continuous. Assume*

$$\sup_j \mathbf{V}^{\mathcal{B}}_{W_j}(0;\sigma) < \infty,$$

$$\|W_j(r,t) - W(r,t)\| + \|F_j(r,t) - F(r,t)\| \to 0, \quad j \to \infty,$$

uniformly in $(r,t) \in \bar{\Delta}_{\rho,\sigma}$. Then

$$W_j \star F_j \to W \star F, \quad j \to \infty, \text{ uniformly on } \bar{\Delta}_{\rho,\sigma}.$$

Proof. If the statement does not hold, we find sequences $r_j \leq t_j$ and $\epsilon > 0$ such that

$$\|(W_j \star F_j)(r_j,t_j) - (W \star F)(r_j,t_j)\| \geq \epsilon \quad \forall\, j. \tag{6.5}$$

$W_j \star F_j$ and $W \star F$ are continuous by Proposition 4.4 and can continuously be extended to $[\rho,\sigma]^2$ by setting it 0 outside of $\bar{\Delta}_{\rho,\sigma}$. After choosing a subsequence, we can assume that $r_j \to r, t_j \to t,\ j \to \infty$ and $(W \star F)(r_j,s) \to (W \star F)(r,s)$ uniformly in $s \in [\rho,\sigma]$. Apply Proposition 3.5 with $x_k(s) = W_k(r_k,s)$, $y_k(s) = F(s,t_k)$ obtaining a contradiction to (6.5).

After having shown (6.2) for V_j, V we finally turn to (6.1) for U_j, U. Here we use $U_j = U_j^\circ + V_j \star U_j^\circ$ and $U = U^\circ + V \star U^\circ$. We again employ Lemma 6.2, but this time with $\mathcal{X} = \mathcal{L}(X), \mathcal{Y} = \mathcal{Z} = X$ and $F_j(r,t) = U_j^\circ(r,t)x$, $F(r,t) = U^\circ(r,t)x$ for fixed, but arbitrary $x \in X$.

7. Isotone operator kernels on abstract M-spaces and order preserving perturbations of evolutionary systems. Integration with respect to functions of bounded semi-variation is less satisfactory than with respect to functions of bounded variation. The reason why we go through this is an important class of operator kernels on a Banach space of continuous functions for which bounded semi-variation is automatically given, whereas bounded variation is not.

Let Ω be a locally compact Hausdorff space and let $X = C_0(\Omega)$, be the closure of the space of continuous functions with compact support in the Banach space of bounded continuous functions, with all three spaces being endowed with the supremum norm. Somewhat loosely we call $C_0(\Omega)$ the space of continuous functions on Ω that vanish at infinity.

The dual space of X is as usually identified with the Banach space $M(\Omega)$ of regular complex Borel measures on Ω with the total variation as norm. X is also a Banach lattice with the natural ordering,

$$x \le y \iff x(\theta) \le y(\theta) \quad \forall \theta \in \Omega.$$

Let X_+ denote the cone of non-negative elements in X. The lattice structure is inherited by X^* with the cone X_+^* of non-negative functionals,

$$x^* \in X_+^* \iff \quad \langle x, x^* \rangle \ge 0 \quad \forall x \in X_+.$$

The isomorphism between X^* and $M(\Omega)$ is also order-preserving, i.e., non-negative functionals correspond to non-negative regular Borel measures. We note that, for x^* corresponding to $\xi \in M(\Omega)$,

$$\|x^*\| = \xi(\Omega), \quad x^* \in X_+^*. \tag{7.1}$$

More generally, let X be an abstract M-space, i.e., X is a Banach lattice whose norm satisfies

$$\|x \vee y\| = \|x\| \vee \|y\|, \quad x, y \in X_+,$$

with $x \vee y$ denoting the supremum of two elements in a Banach lattice. The dual space X^* of an abstract M-space is an abstract L-space, i.e.,

$$\|x^* + y^*\| = \|x^*\| + \|y^*\|, \quad x^*, y^* \in X_+^*.$$

See (Schaefer,1974, Chapter II). We will need this relation in the following form:

$$\|x^* - y^*\| = \|x^*\| - \|y^*\|, \quad 0 \le y^* \le x^*, \quad x^*, y^* \in X^*. \tag{7.2}$$

We have the following surprising result:

Proposition 7.1. *Let X be an abstract M-space and Z be a Banach lattice. Let $L : [r, t] \to \mathcal{L}(X, Z)$ be nondecreasing, i.e.,*

$$(L(s_2) - L(s_1))(X_+) \subseteq Z_+ \quad \textit{whenever} \quad r \le s_1 \le s_2 \le t.$$

Then L is of bounded semi-variation and

$$\mathbf{v}_L(r; t) \le \|L(t) - L(r)\|.$$

Proof. Guided by Proposition 3.11 (with the bilinear triple $(\mathcal{L}(X, Z), X, Z)$) we consider $\mathbf{v}^\bullet_{L(\cdot)z^*}(r; t)$ for $\|z^*\| \le 1$. Let $r = t_0 < \cdots t_{n+1} = t$ be a partition of $[r, t]$. As the dual of a non-negative operator is also non-negative we have

$$\begin{aligned} &\sum_{j=0}^{n} \left\| \Big(L^*(t_{j+1}) - L^*(t_j)\Big) z^* \right\| \\ &\le \sum_{j=0}^{n} \left\| \Big(L^*(t_{j+1}) - L^*(t_j)\Big) |z^*| \ \right\| = \sum_{j=0}^{n} \left\| L^*(t_{j+1})|z^*| - L^*(t_j)|z^*| \ \right\|. \end{aligned}$$

As $L^*(t_{j+1})|z^*| \geq L^*(t_j)|z^*|$ and these two elements belong to the abstract L-space X^*, we can use (7.2) and continue by

$$\leq \sum_{j=0}^{n} \left(\left\| L^*(t_{j+1})|z^*| \right\| - \left\| L^*(t_j)|z^*| \right\| \right) = \left\| L^*(t)|z^*| \right\| - \left\| L^*(r)|z^*| \right\|$$
$$= \left\| \left(L^*(t) - L^*(r) \right) |z^*| \right\| \leq \|L^*(t) - L^*(r)\| = \|L(t) - L(r)\|,$$

because, in the Banach lattice Z^*, $\|z^*\| = \||z^*|\|$. The assertion now follows from Proposition 3.11.

Definition 7.2. A Volterra kernel $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}_+(X)$ or $V : \Delta_{\rho,\tau} \to \mathcal{L}_+(X)$ is called an *isotone Volterra kernel* if $V(r,t)$ is monotone non-decreasing in t, i.e.,

$$V(r,s) \leq V(r,t) \quad \text{whenever} \quad 0 \leq r \leq s \leq t.$$

The following is now an easy consequence of Theorem 7.1.

Theorem 7.3. *Any isotone Volterra kernel on an abstract M-space is a Volterra-Stieltjes operator kernel. Any continuous isotone Volterra kernel vanishing on the diagonal on an abstract M-space is regular.*

We conclude this section by showing that perturbation of backward evolutionary systems by step responses that are isotone Volterra kernels is positivity-preserving.

The following lemma is a straightforward consequence of the definition of the Stieltjes integral and is left to the reader.

Lemma 7.4. *Let $V : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}_+(X)$ be an isotone Volterra kernel and let $U : \bar{\Delta}_{\rho,\sigma} \to \mathcal{L}_+(X)$ be a strongly continuous operator family.*

a) *Then $V \star U$ is a family of non-negative operators.*

b) *If V, U are continuous isotone Volterra kernels vanishing on the diagonal, so is $V \star U$.*

Using the series expansion (4.11) and induction this leads to the following result.

Theorem 7.5. *Let X be an abstract M-space. If V_0 is a continuous isotone (local) Volterra kernel on X vanishing on the diagonal, so is its resolvent kernel V.*

The variation-of-constants formula $U = U_0 + V \star U_0$ provides a perturbation result for evolutionary systems in which positivity is preserved. We notice that a non-negative step response for a non-negative backward evolutionary system is an isotone kernel automatically.

Corollary 7.6. *Let X be an abstract M-space and $U_0 : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ be a strongly continuous backward evolutionary system of non-negative operators and let $V_0 : \Delta_{\rho,\tau} \to \mathcal{L}(X)$ be a continuous step response for U_0 consisting of non-negative operators as well. Then the perturbation of U_0 by V_0 leads to a strongly continuous evolutionary system U that also consists of non-negative operators.*

8. Applications to physiologically structured population models. Consider a population the individuals of which are characterized by an n-dimensional state vector (i-state). The set $\Omega \subset \mathbf{R}^n$ of feasible i-states is called the i-state space. Traditional modeling of deterministic dynamics of such structured populations starts by prescribing the i-state specific *rates* of change of i-state ('growth'), of dying and of giving birth, and the distribution of i-states of neonates. Lifting the model from the individual level to the population level is a matter of careful book-keeping and leads to a population balance equation, which generally has the form of a first order hyperbolic partial (functional) differential equation. We refer the reader to the book by Metz and Diekmann (1986) for a general account of modeling structured populations as well as a wealth of examples and also mention the paper by Tucker and Zimmerman (1988).

Trying to analyze the PDE describing population balance, one faces all the difficulties mentioned in the introduction. In this section we shall formulate the population model in 'cumulative' terms as opposed to rates and show that with this formulation structured population dynamics fits nicely into the abstract framework developed in the previous sections.

We distinguish between two processes on the individual level: (i) i-state change and survival, and (ii) reproduction.

The change of i-state is modeled by prescribing a function $Y(t, s, y)$, giving the i-state at time t of an individual who had i-state y at time s provided the individual has not died in the mean time. The *survival function* $\mathcal{F}(t, s, y)$ gives the probability that an individual who had i-state y at time s is still alive at time t. The interpretations require that Y should have the semigroup property

$$Y(t, r, y) = Y(t, s, Y(s, r, y)), \quad Y(s, s, y) = y \tag{8.1}$$

while $\mathcal{F}$ should satisfy the consistency relation

$$\mathcal{F}(t, r, y) = \mathcal{F}(t, s, Y(s, r, y))\mathcal{F}(s, r, y), \quad \mathcal{F}(s, s, y) = 1. \tag{8.2}$$

Reproduction is modeled by prescribing a *reproduction kernel* Λ defined as follows: $\Lambda(t, s, y)(\omega)$ is the expected total number of direct offspring (i.e., children but not grand children, great grand children, etc.) with state-at-birth in the (measurable) subset ω of Ω, of an individual who had i-state y at time s, in the time-interval $[s, t]$. Λ is thus a function from $\Delta^*_{\tau,\rho} \times \Omega$ to $M(\Omega)$. Here $M(\Omega)$ is the Banach space of regular Borel measure on Ω, which we identify with the dual space of $C_0(\Omega)$, see Section 7. Notice that in the definition of Λ we do not condition on survival of the individual till time t. Consequently, the appropriate consistency relation is

$$\Lambda(t, r, y) = \Lambda(s, r, y) + \Lambda(t, s, Y(s, r, y))\mathcal{F}(s, r, y). \tag{8.3}$$

The model ingredients on the individual level are Y, $\mathcal{F}$ and Λ. Next we use these to define forward and backward evolutionary systems and corresponding cumulative outputs and step responses.

The unperturbed systems describe i-state change and survival, disregarding reproduction. Using the above description of i-state change and survival we can immediately write down the time evolution on the population level of a sterile population. By the state of the population (p-state) we understand the (unnormalized) distribution of i-states. Let $\phi \in M(\Omega)$ be the p-state at time s. Then the *generation development operator family* U_0^* : $\Delta^*_{\tau,\rho} \to \mathcal{L}(M(\Omega))$ defined by

$$(U_0^*(t, s)\phi)(\omega) = \int_\Omega \chi_\omega(Y(t, s, y))\mathcal{F}(t, s, y)\phi(dy), \tag{8.4}$$

where χ denotes the characteristic or indicator function, gives the p-state at time t of the zeroth generation (i.e., those present at time s). The consistency relations (8.1) and (8.2) guarantee that U_0^* is a forward evolutionary system on $M(\Omega)$. It is the (formal) adjoint of the backward evolutionary system $U_0 : \Delta_{\rho,\tau} \to \mathcal{L}(C_0(\Omega))$ defined by

$$(U_0(s,t)f)(y) = f(Y(t,s,y))\mathcal{F}(t,s,y), \quad f \in C_0(\Omega). \tag{8.5}$$

Next we take reproduction into account. The (cumulative) *direct offspring operator family* $V_0^* : \Delta_{\tau,\rho}^* \to \mathcal{L}(M(\Omega))$ is defined by

$$(V_0^*(t,s)\phi)(\omega) = \int_\Omega \Lambda(t,s,y)(\omega)\phi(dy). \tag{8.6}$$

If, at time s, the p-state of the population is given by the measure ϕ, then $V_0^*(t,s)\phi$ yields the expected cumulative number of direct offspring in the time-interval $[s,t]$, as distributed with respect to the i-state at birth. The consistency relation (8.3) shows that V_0^* is a cumulative output for U_0^*. V_0^* is the (formal) adjoint of $V_0 : \Delta_{\rho,\tau} \to \mathcal{L}(C_0(\Omega))$ defined by

$$(V_0(s,t)f)(y) = \int_\Omega f(x)\Lambda(t,s,y)(dx). \tag{8.7}$$

Obviously V_0 is a step response for U_0.

We are interested in the time-evolution of the whole population and not only of the zeroth generation. So let the (cumulative) *total offspring operator* $V^*(t,s)$ be the analogue of $V_0^*(t,s)$ when considering the total clan, i.e., including offspring of offspring, etc.. $(V^*(t,s)\phi)(\omega)$ is thus the expected number of *all* births with i-state at birth in the set $\omega \subset \Omega$, in the time-interval $[s,t]$, given that the p-state at time s was ϕ. Consistency requires that

$$V^*(t,s) = V_0^*(t,s) + \int_s^t V_0^*(t,\sigma)V^*(d\sigma,s) \tag{8.8}$$

since any newborn is either the offspring of an individual already present at time s or of an individual born after time s. Once the equation (8.8) is solved the time-evolution of the population is given by the *population development operator family* $U^* : \Delta_{\tau,\rho}^* \to \mathcal{L}(M(\Omega))$ defined by

$$U^*(t,s) = U_0^*(t,s) + \int_s^t U_0^*(t,\sigma)V^*(d\sigma,s). \tag{8.9}$$

The (pre) adjoint equations of (8.8) and (8.9) are of course

$$V(s,t) = V_0(s,t) + \int_s^t V(s,d\sigma)V_0(\sigma,t) \tag{8.10}$$

and

$$U(s,t) = U_0(s,t) + \int_s^t V(s,d\sigma)U_0(\sigma,t), \tag{8.11}$$

respectively.

We are now exactly in the situation described in Theorem 5.2. To solve the population problem, that is, to find the population development operator, we have to impose conditions on the model ingredients Y, $\mathcal{F}$ and Λ that guarantee that U_0 is a strongly continuous backward evolutionary system and that V_0 is a regular step response for U_0. Since U_0 and V_0 act on the abstract M-space $C_0(\Omega)$ we can use the results of Section 7.

Theorem 8.1. *Let* $Y : \Delta^*_{\tau,\rho} \times \Omega \to \Omega$, $\mathcal{F} : \Delta^*_{\tau,\rho} \times \Omega \to [0, 1]$ *and* $\Lambda : \Delta^*_{\tau,\rho} \times \Omega \to M(\Omega)_+$ *satisfy the relations* (8.1)–(8.3) *and assume that they are continuous in* (t, r) *on* $\Delta^*_{\tau,\rho}$, *uniformly in* $y \in \Omega$ *and continuous in* y *on* Ω *for each* $(t, r) \in \Delta^*_{\tau,\rho}$. *Assume further that for every* $(t, s) \in \Delta^*_{\tau,\rho}$ *and every Borel set* $\omega \in \Omega$ *the function* $\Lambda(t, s, \cdot)(\omega)$ *belongs to* $C_0(\Omega)$ *and that for any* $(t, s) \in \Delta^*_{\tau,\rho}$ *either*

(i) *the function* $\mathcal{F}(t, s, \cdot)$ *belongs to* $C_0(\Omega)$, *or*

(ii) *for any compact subset* K *of* Ω *there exists another compact subset* $\tilde{K}$ *of* Ω *such that* $Y(t, s, y) \in \Omega \setminus K$ *whenever* $y \in \Omega \setminus \tilde{K}$.

Then U_0 *defined by* (8.5) *is a strongly continuous backward evolutionary system and* V_0 *defined by* (8.7) *is a regular step response for* U_0 *on* $C_0(\Omega)$. *The perturbation of* U_0 *by* V_0 *leads to a strongly continuous evolutionary system* U.

Proof. The continuity assumptions imply that U_0 is strongly continuous and that V_0 is continuous with respect to the uniform operator topology. The assumptions that $\Lambda(t, s, \cdot)(\omega)$ belongs to $C_0(\Omega)$ and either (i) or (ii) guarantee that $U_0(s, t)$ and $V_0(s, t)$ both belong to $\mathcal{L}(C_0(\Omega))$. As pointed out above, the relations (8.1) – (8.3) guarantee that U_0 is a backward evolutionary system and that V_0 is a step response for U_0. The positivity of $\mathcal{F}$ and Λ imply that U_0 and V_0 consist of positive operators. The result now follows from Corollary 7.6.

It is also easy to see that the solution $U(s, t)$ depends continuously on the model ingredients Y, $\mathcal{F}$ and Λ. This is the content of the following theorem.

Theorem 8.2. *Let* Y, Y_j, $\mathcal{F}$, $\mathcal{F}_j$ *and* Λ, Λ_j *satisfy the hypotheses of Theorem* 8.1 *and let* U°, U_j°, V°, V_j° *be the corresponding evolutionary systems and step responses according to* (8.5) *and* (8.7). *If* $Y_j \to Y$, $\mathcal{F}_j \to \mathcal{F}$ *and* $\Lambda_j \to \Lambda$ *uniformly on* $\Delta^*_{\tau,\rho} \times \Omega$ *as* $j \to \infty$, *then the assumptions of* Theorem 6.1 *are satisfied.*

Proof. This is a consequence of Proposition 3.5 and the fact that the variation of the non-increasing functions $\mathcal{F}_j(\cdot, r, y)$, $\mathcal{F}(\cdot, r, y)$ are bounded by 1.

9. A size structured population model. In this section we illustrate the advantages of the cumulative formulation by an example of a size structured population model. The first step in modeling structured populations is usually a verbal description of assumptions about growth, survival and reproduction. The verbal formulation of the model assumptions can be translated in different ways into consistent mathematical formulation. At first this mathematical formulation is formal. It is, subsequently, the task of the analyst to make this formulation precise. Below we give an example, where this task turns out to be quite awkward if one uses a PDE formulation, whereas it is relatively easy if one uses the cumulative approach.

In our model we assume that individuals are fully characterized by the number $y \in \Omega := [y_B, \infty)$ called "size". All individuals are assumed to have the same size y_B at birth. If modeling is based on rates one obtains the following well-known problem for the size density $n(t, y)$ of the population.

$$
\begin{aligned}
&\frac{\partial}{\partial t}n(t, y) + \frac{\partial}{\partial y}(g(t, y)n(t, y)) = -\mu(t, y)n(t, y), \quad y_B < y < \infty, \quad s < t < \tau,\\
&g(t, y_B)n(t, y_B) = \int_{y_B}^{\infty} \beta(t, y)n(t, y)dy, \quad s < t < \tau,\\
&n(s, y) = \phi(y), \qquad (9.1)
\end{aligned}
$$

where $g(t, y)$ is the growth rate of an individual of size y at time t, $\mu(t, y)$ and $\beta(t, y)$ are the size specific death and birth rates at time t and ϕ is the initial size distribution.

We assume that individuals do not reproduce until they reach a certain threshold size y_A (A for adult). The birth rate β therefore has the form

$$\beta(t, y) = \begin{cases} 0, & \text{if } \; y_B \le y < y_A \\ \beta_0(t, y), & \text{if } \; y_A < y. \end{cases} \tag{9.2}$$

Observe that we do not require β to be continuous at $y = y_A$. In fact, in concrete models it is often advantageous to allow for discontinuous rates, because in this way models remain parameter sparse — continuous rates often require too many parameters. Notice also that we do not specify the value of $\beta(t, y)$ at $y = y_A$. The reason is that there is usually no biological indication from observations or inherent logic how the birth rate should be defined at $y = y_A$.

The model (9.1) has the property that even if the initial p-state is an absolutely continuous measure with an arbitrarily smooth density $n(s, \cdot) = \phi$, the solution need not remain absolutely continuous. This typically happens when individuals are born under conditions when they cannot grow ($g(t, y_B) = 0$ for an extended time) and so a cohort (mathematically represented by a Dirac δ-measure) at the birth size y_B is formed. The noninvariance of the space of absolutely continuous measures is no problem *per se*, solutions can still be defined in a weak*-sense.

If cohorts always cross the critical size y_A with positive speed (i.e., if $g(t, y_A)$ is always strictly positive), there are still no problems with the model (9.1), see Calsina and Saldana (preprint) who even deal with the case that g depends on total population size. But suppose a cohort has been formed at y_B as described above. If $g(t, y)$ becomes positive this cohort starts moving to the right. No smoothening will occur and eventually the cohort reaches the threshold size y_A. If $g(t, y)$ becomes zero again exactly when the cohort reaches y_A the problem is not well-posed. The solution depends on the specific choice of the value of $\beta(t, y_A)$. In nonlinear problems where β does not depend explicitly on t but only through the environment, which in turn changes as a consequence of the activity of the population, the situation is even worse: then uniqueness may fail if cohorts stop at the critical size, see Thieme (1988).

Whether or not well-posedness will hold is, as seen by the example above, determined by the global and combined effect of the rates. These problems are completely hidden in the PDE formulation (9.1) and only a complete mathematical analysis of that model will reveal whether or not the model is "good", that is, well-posed. In the cumulative formulation the basic ingredients Y, $\mathcal{F}$ and Λ already reflect such global and combined effects. The test of well-posedness is therefore part of the model-building and not of the subsequent analysis of the model. Assume now that the model is originally, say on the basis of mechanistic arguments and physiological measurements, formulated in terms of rates as in (9.1). The model can then be recast in cumulative terms by constructing Y, $\mathcal{F}$ and Λ from the given rates. This reformulation is not simply a matter of mathematical convenience. It introduces a phase between model-building and analysis and whether the model is good or not is tested in this phase. To illustrate this we next formulate the population model in cumulative terms.

To be specific we make the following smoothness assumptions about the rates.

(H_g) $g : [\rho, \tau] \times [y_B, \infty) \to \mathbf{R}$ is nonnegative, bounded and continuous, and satisfies the following Lipschitz condition: For each $b > y_B$ there exists an $L > 0$ such that

$$|g(t, y) - g(t, \tilde{y})| \le L|y - \tilde{y}|, \qquad \rho \le t \le \tau, \;\; y_B \le y, \tilde{y} \le b. \tag{9.3}$$

(H_μ) $\mu : [\rho, \tau] \times [y_B, \infty) \to \mathbf{R}$ is nonnegative, Borel measurable with respect to its first argument and continuous with respect to its second argument.

(H_β) $\beta : [\rho, \tau] \times [y_B, \infty) \to \mathbf{R}$ is nonnegative, bounded and Borel measurable and

$$\lim_{y\to\infty} \beta(s, y) = 0, \qquad \text{uniformly for } \rho \le s \le \tau. \tag{9.4}$$

The function $Y(t, s, y)$ describing i-state change is the unique solution to the initial value problem

$$\begin{aligned} \frac{d}{dt} Y(t, s, y) &= g(t, Y(t, s, y)), \quad s < t \le \tau, \\ Y(s, s, y) &= y, \end{aligned} \tag{9.5}$$

and the survival function is given by

$$\mathcal{F}(t, s, y) = e^{-\int_s^t \mu(\sigma, Y(\sigma, s, y))d\sigma}. \tag{9.6}$$

The reproduction kernel has the form

$$\Lambda(t, s, y) = \int_s^t \beta\,(\sigma, Y(\sigma, s, y))\,\mathcal{F}(\sigma, s, y)d\sigma\;\delta_{y_B}, \quad \rho \le t \le \tau, \quad y_B \le y. \tag{9.7}$$

Observe that the range of $\Lambda(t, s, y)$ is one dimensional. We shall also make use of the corresponding *conditional* (on survival) reproduction kernel λ defined by

$$\lambda(t, s, y) := \int_s^t \beta(\sigma, Y(\sigma, s, y))d\sigma. \tag{9.8}$$

In the sequel we shall always denote by Y, $\mathcal{F}$, Λ and λ the functions corresponding to g, μ and β through formulae (9.5)–(9.8). When we consider families of functions g, μ and β indexed in a certain way, the corresponding functions Y, $\mathcal{F}$, Λ and λ will without explicit mentioning be indexed in the same way, that is, Y_j corresponds to g_j through (9.5) etc.

We want to apply Theorem 8.1 to deduce the well-posedness of the population problem. It is obvious that the functions Y, $\mathcal{F}$ and Λ satisfy the consistency relations (8.1)–(8.3). The hypotheses (H_g) and (H_μ) clearly imply that Y and $\mathcal{F}$ have the continuity properties required in Theorem 8.1. Since g is assumed to be non-negative one has $Y(t, s, y) \ge y$ for all $(t, s) \in \Delta^*_{\tau,\rho}$ and therefore condition (ii) of Theorem 8.1 is automatically satisfied. By assumption (9.4) one has

$$\lim_{y\to\infty} \Lambda(t, s, y)(\omega) = 0$$

for all $(t, s) \in \Delta^*_{\tau,\rho}$ and all Borel sets $\omega \subset [y_B, \infty)$ and therefore the crucial condition to check is the continuity of the reproduction kernel Λ. It is clear from formula (9.7) that continuity properties of Λ are consequences of combined properties of β and g. The following notion of g-continuity is appropriate to describe this combined effect. First we introduce the symbol

$$BM = BM([\rho, \tau] \times [y_B, \infty))$$

to denote the space of real-valued bounded Borel measurable functions β on $[\rho, \tau]\times[y_B, \infty)$.

Definition 9.1. The function $\beta \in BM$ is called *g-continuous* if it has the following two properties:

(i) For $s \in [\sigma, \tau]$, $y \in [y_B, \infty)$:

$$g(s, y) = 0 \quad \Longrightarrow \quad \lim_{\eta\to y} \beta(s, \eta) = \beta(s, y). \tag{9.9}$$

(ii) For any $\delta > 0$, $b > y_B$, there is a set $N_\delta \subseteq [y_B, b]$ such that the Lebesgue measure of $[y_B, b] \setminus N_\delta$ is less than δ and for $y \in N_\delta$

$$\lim_{\eta\to y} \beta(s, \eta) = \beta(s, y) \quad \text{uniformly for } \rho \le s \le \tau. \tag{9.10}$$

Notice that in the time independent case, where β does not depend on s, condition (ii) is automatically satisfied by Lusin's theorem.

Lemma 9.2. *Let $\beta \in BM$ be g-continuous. Then $\lambda(\cdot, r, y)$ is globally Lipschitz continuous for each $r \in [\rho, \tau)$, $y \in [y_B, \infty)$ and $\lambda(t, \cdot, \cdot)$ is continuous for each $t \in (\rho, \sigma]$.*

Proof. The Lipschitz continuity of $\lambda(t, r, y)$ in t follows from the boundedness of β. To show the continuity of $\lambda(t, r, y)$ in r and y consider sequences $r_j \to r, y_j \to y, \ j \to \infty$. Since, if $t = r$,

$$\lambda(t, r_j, y_j) - \lambda(t, r, y) = \int_{r_j}^{r} \beta(s, Y(s, r_j, y_j))ds \to 0, \qquad j \to \infty$$

we can assume that $t > r, r_j$.

From Fatou's lemma and the boundedness of β we obtain

$$\limsup_{j\to\infty} |\lambda(t, r_j, y_j) - \lambda(t, r, y)| \le \int_r^t \limsup_{j\to\infty} |\beta(s, Y(s, r_j, y_j)) - \beta(s, Y(s, r, y))|ds.$$

As β is g-continuous we obtain

$$\begin{aligned}&\limsup_{j\to\infty} |\lambda(t, r_j, y_j) - \lambda(t, r, y)| \\ &\le \int_r^t \limsup_{j\to\infty} |\beta(s, Y(s, r_j, y_j)) - \beta(s, Y(s, r, y))| \ \chi_{\{s;|g(s,Y(s,r,y))|>0\}}(s)ds.\end{aligned}$$

As the integrand is bounded and the measure of the set $\{s \in [r, t]; |g(s, Y(s, r, y))| > \delta\}$ converges to the measure of the set $\{s \in [r, t]; |g(s, Y(s, r, y))| > 0\}$ as $\delta \searrow 0$, we find for any $\epsilon > 0$ some $\delta > 0$ such that

$$\begin{aligned}&\limsup_{j\to\infty} |\lambda(t, r_j, y_j) - \lambda(t, r, y)| \\ &\le \int_r^t \limsup_{j\to\infty} |\beta(s, Y(s, r_j, y_j)) - \beta(s, Y(s, r, y))|\chi_{\{s;|g(s,Y(s,r,y))|>\delta\}}(s)ds + \epsilon \\ &\le \int_r^t \limsup_{j\to\infty} \sup_{r\le p\le t} \left|\beta\big(p, Y(s, r, y)) + Y(p, r_j, y_j) - Y(p, r, y)\big) - \beta\big(p, Y(s, r, y)\big)\right| \\ &\quad \times \frac{1}{\delta}|g(s, Y(s, r, y))|ds + \epsilon.\end{aligned}$$

With the substitution $\zeta = Y(s, r, y)$ we obtain

$$\begin{aligned}&\limsup_{j\to\infty} |\lambda(t, r_j, y_j) - \lambda(t, r, y)| \\ &\le \frac{1}{\delta} \int_{Y([r,t],r,y)} \limsup_{j\to\infty} \sup_{r\le p\le t} \left|\beta\Big(p, \zeta + Y(p, r_j, y_j) - Y(p, r, y)\Big) - \beta(p, \zeta)\right| d\zeta + \epsilon.\end{aligned}$$

Here $Y([r, t], s, y)$ denotes the set $\{Y(p, s, y); r \le p \le t\}$. Let $b > y_B$ be such that $Y(t, s, y) \le b$ for $r \le s \le t$. Then we use the property formulated in Definition 9.1 (ii) for this b. Using again that the integrand is bounded we find a set N_ϵ such that

$$\begin{aligned}&\limsup_{j\to\infty} |\lambda(t, r_j, y_j) - \lambda(t, r, y)| \\ &\le \frac{1}{\delta} \int_{Y([r,t],r,y)\cap N_\epsilon} \limsup_{j\to\infty} \sup_{r\le p\le t} \left|\beta\big(p, \zeta + Y(p, r_j, y_j) - Y(p, r, y)\big) - \beta(p, \zeta)\right| d\zeta + 2\epsilon \\ &\le 2\epsilon\end{aligned}$$

because $Y(p, r_j, y_j) - Y(p, r, y) \to 0$ for $j \to \infty$ uniformly in $\rho \le p \le \tau$ and (9.10) holds.

Corollary 9.3. *Let the conditions* (H_g), (H_μ), (H_β) *hold and assume that* β *is g-continuous. Then all the assumptions of Theorem* 8.1 *hold.*

Proof. The only thing that remains to be checked is that $\Lambda(t, s, y)$ defined by (9.7) is continuous in (t, s) on $\Delta^*_{\tau,\rho}$ uniformly in $y \in [y_B, \infty)$ and continuous in y on $[y_B, \infty)$ for each $(t, s) \in \Delta^*_{\tau,\rho}$. But this follows readily from the preceding lemma, the continuity of $\mathcal{F}$ and the fact that $\mathcal{F}$ is less than one.

Next we turn to the question of the influence on Λ of a change of the birth rate on a set of Lebesgue measure zero. As we have seen in the example above such a change may indeed have the undesirable effect of altering Λ if g happens to be zero for an extended period of time in the null-set where β is modified. We make the following definition.

Definition 9.4. $\beta_1, \beta_2 \in BM$ are called g-equivalent if the following two conditions hold:
(i) For any $s \in [\sigma, \tau]$, $y \in [y_B, \infty)$,

$$g(s, y) = 0 \implies \beta_1(s, y) = \beta_2(s, y). \tag{9.11}$$

(ii) There exists a subset N of $[y_B, \infty)$ of Lebesgue measure 0 such that

$$\beta_1(s, y) = \beta_2(s, y) \quad \forall s \in [\rho, \tau],\ y \in \Omega \setminus N. \tag{9.12}$$

Note that g-equivalence is indeed an equivalence relation. The name "g-equivalent" is motivated by the following result.

Lemma 9.5. *If* $\beta_1 \in BM$ *and* $\beta_2 \in BM$ *are g-equivalent, then* $\lambda_1 = \lambda_2$.

Proof. Let N be the set of y for which β_1 and β_2 are different. Let c be the bound of $|\beta_1(s, y) - \beta_2(s, y)|$ for $r \le s \le t$. Then, by (9.11) and (9.12),

$$\begin{aligned}
&|\lambda_1(t, r, y) - \lambda_2(t, r, y)| \le c \int_r^t \chi_N(Y(s, r, y))\chi_{\{s;|g(s,Y(s,r,y))|>0\}}(s)ds \\
&= \lim_{j\to\infty} c \int_r^t \chi_N(Y(s, r, y))\chi_{\{s;|g(s,Y(s,r,y))|>1/j\}}(s)ds \\
&\le \lim_{j\to\infty} c \int_r^t \chi_N(Y(s, r, y)) j |g(s, Y(s, r, y))| ds \\
&= \lim_{j\to\infty} c \int_r^t \chi_N(Y(s, r, y)) j |g(s, Y(s, r, y))| ds = \lim_{j\to\infty} cj \int_{Y([r,t],s,y)} \chi_N(\eta)d\eta = 0.
\end{aligned}$$

Here $Y([r, t], s, y)$ denotes the set $\{Y(p, s, y); r \le p \le t\}$.

Corollary 9.6. *If* $\beta_1 \in BM$ *and* $\beta_2 \in BM$ *are g-equivalent, then* $\Lambda_1 = \Lambda_2$.

Proof. Obvious by the preceding lemma and the fact that $\mathcal{F}$ is less than one.

Let us now return to the population problem with birth rate given by (9.2). Corollary 9.3 and Corollary 9.6 show us what extra assumptions we have to impose on β_0 in order to get a well-posed problem that does not depend on the specific value of $\beta(s, y_A)$. Indeed, if β_0 is continuous and satisfies

$$\lim_{y\to\infty} \beta_0(s, y) = 0, \quad \text{uniformly for } s \in [\rho, \tau],$$

and if for any $s \in [\rho, \tau]$

$$g(s, y_A) = 0 \implies \beta_0(s, y_A) = 0, \tag{9.13}$$

then, whatever value between 0 and $\beta_0(t, y_A)$ we choose for $\beta(t, y_A)$, the reproduction kernel Λ will not depend on this choice. Moreover, for any of these choices, β is g-continuous. For a concrete example concerning waterflea dynamics and a biological interpretation of (9.13) in terms of individual energy budgeting, see Thieme (1988), Section 2 and Section 5, see also Kooijman (1993).

As mentioned above, the reason for modeling birth rates as functions with jumps is to keep the concrete expression for the birth rate parameter-sparse. It is therefore important to reflect upon the question when exactly it is justified to replace a steep (but still continuous) increase of the birth rate around a threshold value y_A by a jump at y_A. This is obviously a question of continuous dependence of Λ on g and β.

Definition 9.7. Let β and β_j, $j \in \mathbf{N}$, be functions in BM. Then β_j is called g-convergent towards β for $j \to \infty$ if and only if the following two properties hold:

(i) If $\rho \leq s \leq \tau$ and $y, y_j \in \Omega$, $y_j \to y$ for $j \to \infty$, then

$$g(s, y) = 0 \Longrightarrow \lim_{j\to\infty} \beta_j(s, y_j) = \beta(s, y).$$

(ii) For any $\delta > 0$, $b > y_B$ there exists a set $N_\delta \subseteq [y_B, b]$ such that the Lebesgue measure of $[y_B, b] \setminus N_\delta$ is less than δ and, for $y \in N_\delta$,

$$\beta_j(s, y+h) \to \beta(s, y) \text{ for } j \to \infty,\ h \to 0, \text{ uniformly in } s \in [\rho, \tau].$$

Lemma 9.8. *Assume that g_j, g are bounded continuous real-valued functions on $[\rho, \tau] \times \Omega$ and that*

$$g_j \to g \text{ for } j \to \infty \text{ uniformly on } [\rho, \tau] \times \Omega.$$

Moreover let β, β_j, $j \in \mathbf{N}$, be functions in BM with a uniform bound, β_j g-convergent towards β for $j \to \infty$, and let β and β_j be g and g_j continuous respectively. Then the following holds.

(a) *$Y_j \to Y$ uniformly on $\Delta^*_{\tau,\rho} \times \Omega$ as $j \to \infty$*

(b) *$\lambda_j \to \lambda$ uniformly on $\Delta^*_{\tau,\rho} \times [y_B, b]$ as $j \to \infty$ for any $b > y_B$.*

(c) *If, in addition,*

$$\beta_j(s, y), \beta(s, y) \to 0 \text{ for } y \to \infty \text{ uniformly in } j \in \mathbf{N}, s \in [\rho, \tau]$$

then $\lambda_j \to \lambda$ uniformly on $\Delta^*_{\tau,\rho} \times \Omega$ as $j \to \infty$.

Proof. (a) This is a standard result in the theory of ordinary differential equations.

(b) Assume that the assertion does not hold. Then we find sequences $\rho \leq r_j \leq t_j \leq \tau$ and $y_j \in [0, b]$ and some $\epsilon > 0$ such that

$$|\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| > \epsilon \quad \text{for } j \to \infty. \tag{9.14}$$

We can assume that $r_j \to r$, $t_j \to t$, $y_j \to y$ for some $0 \leq r \leq t \leq t \leq \sigma, 0 \leq y \leq b$.

As λ is continuous by Lemma 9.2 we have

$$\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \leq \limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t, r, y)|.$$

From Fatou's lemma and the boundedness of β we obtain

$$\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \leq \int_r^t \limsup_{j\to\infty} |\beta_j(s, Y_j(s, r_j, y_j)) - \beta(s, Y(s, r, y))| ds.$$

As β_j is g-convergent to β, we obtain from Definition 9.7 (i) and the fact that $Y_j(s, r_j, y_j) \to Y(s, r, y)$ that

$$\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \leq \int_r^t \limsup_{j\to\infty} |\beta_j(s, Y_j(s, r_j, y_j)) - \beta(s, Y(s, r, y))| \, \chi_{\{s;|g(s,Y(s,r,y))|>0\}}(s)ds.$$

As the integrand is bounded and the measure of the set $\{s \in [r, t]; |g(s, Y(s, r, y))| > \delta\}$ converges to the measure of the set $\{s \in [r, t]; |g(s, Y(s, r, y))| > 0\}$ as $\delta \searrow 0$, we find some $\delta > 0$ such that

$$\begin{aligned}
&\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \\
&\leq \int_r^t \limsup_{j\to\infty} |\beta_j(s, Y_j(s, r_j, y_j)) - \beta(s, Y(s, r, y))| \, \chi_{\{s;|g(s,Y(s,r,y))|>\delta\}}(s)ds + \epsilon/4 \\
&\leq \int_r^t \limsup_{j\to\infty} |\beta_j(s, Y_j(s, r_j, y_j)) - \beta(s, Y(s, r, y))| \frac{1}{\delta} |g(s, Y(s, r, y))| ds + \epsilon/4 \\
&\leq \int_r^t \limsup_{j\to\infty} \sup_{r\leq p\leq t} \Big|\beta_j\Big(p, Y(s, r, y) + Y_j(p, r_j, y_j) - Y(p, r, y)\Big) - \beta\Big(p, Y(s, r, y)\Big)\Big| \\
&\quad \times \frac{1}{\delta} |g(s, Y(s, r, y))| ds + \epsilon/4.
\end{aligned}$$

With the substitution $\zeta = Y(s, r, y)$ we obtain

$$\begin{aligned}
&\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \\
&\leq \frac{1}{\delta} \int_{Y([r,t],r,y)} \limsup_{j\to\infty} \sup_{r\leq p\leq t} \Big|\beta_j\big(p, \zeta + Y_j(p, r_j, y_j) - Y(p, r, y)\big) - \beta\big(p, \zeta\big)\Big| d\zeta + \epsilon/4.
\end{aligned}$$

Let $\tilde{b} > 0$ be such that $Y(t, s, y) \leq \tilde{b}$ for $r \leq s \leq t$. Then we use the property formulated in Definition 9.7 (ii) for this $\tilde{b}$. Using once more that the integrand is bounded we find a set N_ϵ such that the property in Definition 9.7 (ii) holds and

$$\begin{aligned}
&\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \\
&\leq \frac{1}{\delta} \int_{N_\delta} \limsup_{j\to\infty} \sup_{r\leq p\leq t} \Big|\beta_j\Big(p, \zeta + Y_j(p, r_j, y_j) - Y(p, r, y)\Big) - \beta\Big(p, \zeta\Big)\Big| d\zeta + \epsilon/2.
\end{aligned}$$

As $Y_j(p, r_j, y_j) - Y(p, r, y) \to 0$ for $j \to \infty$ uniformly in $p \in [r, t]$ we have by the property in Definition 9.7 (ii) that, for $\zeta \in N_\delta$,

$$\Big|\beta_j\Big(p, \zeta + Y_j(p, r_j, y_j) - Y(p, r, y)\Big) - \beta\Big(p, \zeta\Big)\Big| \to 0$$

for $j \to \infty$ uniformly in $p \in [r, t]$. Hence

$$\limsup_{j\to\infty} |\lambda_j(t_j, r_j, y_j) - \lambda(t_j, r_j, y_j)| \leq \epsilon/2$$

in contradiction to (9.14). This completes the proof of (b)

(c) follows because $\lambda_j(t, r, y), \lambda(t, r, y) \to 0$ for $y \to \infty$ uniformly in $\rho \leq r \leq t \leq \tau$, $j \in \mathbf{N}$.

Corollary 9.9. *Let g, g_j, β, β_j satisfy all the hypotheses of* Lemma 9.8 *and let $\mathcal{F}_j \to \mathcal{F}$ uniformly on $\Delta^*_{\tau,\rho} \times \Omega$ as $j \to \infty$. Then the assumptions of* Theorem 6.1 *are satisfied.*

Proof. This follows immediately from Lemma 9.8 and Theorem 8.2.

Let us finally apply this result to the approximation of steep continuous birth rates by a birth rate with a jump.

Let $\beta_j(t, y)$ be a sequence a continuous bounded functions such that

$$\beta_j(s, y) \to 0, \quad y \to \infty \text{ uniformly in } j \in \mathbf{N}, s \in [\rho, \tau]$$

and

$$\beta_j(s, y) \to 0, \qquad j \to \infty, \quad y \in [y_B, y_A), \text{ uniformly in } s \in [\rho, \tau],$$
$$\beta_j(s, y) \to \beta_0(s, y), \qquad j \to \infty, \quad y > y_A, \text{ uniformly in } s \in [\rho, \tau].$$

It is easy to check that β_j is g-convergent towards β in the sense of Definition 9.7. Applying Corollary 9.9 with $g_j = g$, $\mathcal{F}_j = \mathcal{F}$ we see that our perturbation procedure is robust with respect to replacing a steep (continuous) increase of the birth rate in a neighborhood of a threshold value y_A by a jump at this threshold value provided that the steep increase only occurs if $g(t, y_A)$ is sufficiently different from 0.

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Appendix. In this appendix we give an example of a continuous step response for a strongly continuous semigroup which is not regular. Hence our regularity assumption is not redundant except in the situation of Section 7.

Consider the Banach space $BM([0, \pi] \times [0, 1])$ of real-valued bounded Borel measurable functions with the supremum norm and let X be the closed subspace of those functions x such that $x(a, \theta)$ is continuous in a uniformly in θ, $x(\pi, \theta) = 0$. Then

$$(T(s)x)(a, \theta) = x(s + a, \theta)$$

defines a strongly continuous semigroup on X. Let

$$x_\diamond(a, \theta) = \begin{cases} \theta \sin\left(\frac{a}{\theta}\left[\frac{1}{\theta}\right]\right) & \text{if} \quad 0 \le a \le \theta\pi, \\ 0 & \text{if} \quad a > \theta\pi. \end{cases}$$

Here $[r]$ denotes the largest natural number n with $n \le r$. This has the consequence that $x_\diamond(\theta\pi, \theta) = 0$. Hence $x_\diamond$ is a Borel measurable function on $[0, \infty) \times [0, 1]$, uniformly continuous in a (uniformly in θ), and has support in $[0, \pi] \times [0, 1]$. Moreover $x_\diamond(a, \theta)$ is continuously partially differentiable in $a \in (0, \theta\pi)$. We define

$$(V(s)x)(a, \theta) = x(0, 0)\Big(x_\diamond(s + a, \theta) - x_\diamond(a, \theta)\Big),$$

Notice that $V(s)$ is of the form $\big(T(s) - I\big)C$ and hence a step response for T. One easily calculates that

$$\mathbf{v}_V(0; r) = \sup_{0 \le \theta \le 1} \int_0^{\min(r, \theta\pi)} |\partial_s x_\diamond(s, \theta)|\, ds = \sup_{0 \le \theta \le 1} \theta \int_0^{\min(\frac{r}{\theta}, \pi)[\frac{1}{\theta}]} |\cos(s)|\, ds.$$

Hence, on the one hand,

$$\mathbf{v}_V(0;r) \le \pi, \ \forall r \ge 0,$$

but on the other hand, choosing $\theta = \frac{1}{j}$, $j \in \mathbf{N}$,

$$\mathbf{v}_V(0;r) \ge \liminf_{j\to\infty} \frac{1}{j} \int_0^{j\pi} |\cos(s)|\, ds \ge 2, \ \forall r > 0.$$

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